Addressing the combinatorial and quadratic features of the Nuclear Outages Scheduling Problem with Semidefinite Programming

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Key features of Energy Management Optimization Problems

Characteristics of Energy Management Optimization Problems (EMOPs) :

$\rightarrow$ large-scale problems ;

$\rightarrow$ non-linearities, including combinatorial features ;

$\rightarrow$ subject to random data.

SDP for Energy Management

SDP : a young and promising area of Conic Programming (Boyd and Vandenberghe, 1996; Overton and Wolkowicz, 1997; Wolkowicz et al., 2000) :

- a high modeling power ;
- polynomially solvable (Interior-Point Methods) ;
- a powerful tool for attacking hard combinatorial problems and for taking uncertainty into account.

$\rightarrow$ A. Gorge's PhD :

Semidefinite programming : methods and algorithms for energy management

Objective : explore the potentiality of SDP for EMOPs. 2 lines of research :

1. Relaxation of problems with quadratic and/or combinatorial features (QCQP) ;
2. Taking uncertainty into account.

$\rightarrow$ restitution of the work done in the first line of research.

About SDP

What is a SDP ?

SDP :

$$\left\{\begin{array}{lll} \inf_{X\in\mathbb{S}^n} & A^0 \bullet X & \\ \text{s.t} & A^k \bullet X = b_k, & k = 1,...,m \\ & X \succcurlyeq 0 & \end{array}\right. \leftrightarrow \left\{\begin{array}{ll} \sup_{y\in\mathbb{R}^m} & b^T y \\ \text{s.t} & A^0 - \sum_{k=1}^{m} y_k A^k \succcurlyeq 0 \end{array}\right.$$

with $A \bullet B = \sum_i \sum_j A_{ij} B_{ij}$

$\rightarrow$ extension of Linear Programming to the convex cone of positive semidefinite matrices : $\mathbb{S}^n_+ = \{X \in \mathbb{S}^n : X \succcurlyeq 0\}$.

How to solve a SDP ? Off-the-shelf solvers are available, based for example on interior-point methods (CSDP, DSDP) or bundle methods (ConicBundle) (Benson et al., 2000; Borchers, 1999; Helmberg and Rendl, 2000).

How to build a SDP relaxation

Key problem = Quadratically Constrained Quadratic Program (QCQP) :

$$(QCQP)\left\{\begin{array}{ll} \min & x^T P_0 x + 2p_0^T x + \pi_0 \\ \text{s.t.} & x^T P_j x + 2p_j^T x + \pi_j \leq 0,\ j = 1,\dots,m \end{array}\right.$$

→ generally NP-hard, except in the convex case ($P_j \succcurlyeq 0,\ j = 0,\dots,m$)

→ a wide range of special cases : 0/1-LP, 0/1-QP, SOCP, ...

Reformulation : $x^T P_j x + 2p_j^T x + \pi_j = \begin{pmatrix} \pi_j & p_j^T \\ p_j & P_j \end{pmatrix} \bullet \begin{pmatrix} 1 & x^T \\ x & xx^T \end{pmatrix}$

Standard SDP relaxation :

$$Y = \begin{pmatrix} 1 & x^T \\ x & xx^T \end{pmatrix} \xrightarrow{\text{relaxation}} \left\{\begin{array}{l} Y_{11} = 1 \\ Y \succcurlyeq 0 \end{array}\right. \quad : \quad (SDP)\left\{\begin{array}{ll} \min & \begin{pmatrix} \pi_0 & p_0^T \\ p_0 & P_0 \end{pmatrix} \bullet Y \\ \text{s.t.} & \begin{pmatrix} \pi_j & p_j^T \\ p_j & P_j \end{pmatrix} \bullet Y \leq 0,\ j = 1,\dots,m \\ & Y_{11} = 1,\ Y \succcurlyeq 0 \end{array}\right.$$

→ relaxation of the constraint $\operatorname{rank}(Y) = 1$;

→ generally not tight, except in the convex case ;

→ **need for reinforcement.**

Reinforcement of the standard SDP relaxation

Idea : apply the standard SDP relaxation to an equivalent QCQP :

$$\text{QCQP} \xrightarrow{\text{Equivalent}} \text{QCQP'} \xrightarrow{\text{Relaxation}} \text{SDP}$$

2 possibilities to get an equivalent QCQP :

1. **Reformulating the linear constraints into quadratic constraints ;**

Example : $b \leq a^T x \leq c \Leftrightarrow (a^T x - b)(a^T x - c) \leq 0$

2. **Addition of valid quadratic constraints = *cuts***

Example 1 : $x \in \{0,1\}^n \Rightarrow x_i x_j \geq 0$

Example 2 : $a^T x \leq b, \;\; c^T x \leq d \Rightarrow (a^T x - b)(c^T x - d) \geq 0$

Initial difficulties when applying SDP to EMOPs

Difficulties : solving a SDP :

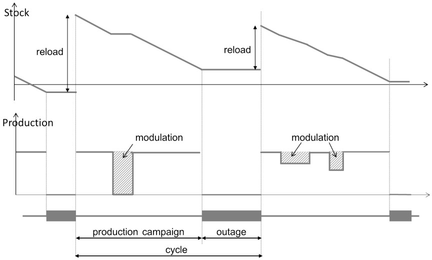
- requires significant computational time (Example : $n = 1000, m = 1000 \rightarrow$ a few hundred seconds) ;
- for large-scale problems : memory capacity exceeded ;
- necessity to define slack variables for inequality constraints (solvers).

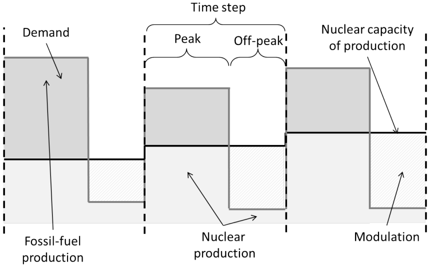
$\rightarrow$ **strong restriction on the problem size, especially with reinforcement**

But EMOPs are huge : necessity to work on the models :

$\rightarrow$ **reduce their size while emphasizing their combinatorial and quadratic features**

Statement of the problem





A first set of simplifications :

1. Power plants are either *on* (during production campaign) or *off* (during outages) : no decrease of the power plant capacity at the end of the production campaign ;
2. No fossil-fuel production while nuclear production not fully used ;
3. Global amount of modulation to satisfy (instead of one value / time step).
4. Deterministic (demand, availability, outages duration, ...)

Our simplified Nuclear Outages Scheduling Problem

Decision variables :

$x_{i,j,t} \in \{0,1\}$	equals 1 if and only if the outage (i,j) starts at t
$m_{i,j} \in [0, M_{i,j}]$	modulation of the cycle (i,j)
$r_{i,j} \in [\underline{R}_{i,j}, \overline{R}_{i,j}]$	reload of the cycle (i,j)

State variables :

c_t	nuclear capacity at time t	linear function of x
p_t	fossil-fuel production at time t	$p_t = \text{peakDemand}_t - c_t$
$t_{i,j}$	beginning date of the outage (i,j)	linear function of x
$f_{i,j}$	ending stock of the cycle (i,j)	linear function of x, m, r and f

Objective and constraints to consider

Nuclear Outages Scheduling Problem :

$$\left\{\begin{array}{lll} \min & \text{nuclear cost}(r, f) + \text{fossil-fuel cost}(p) & \\ \text{s.t.} & c_t + p_t = \text{peakDemand}_t & \text{(peak demand-supply balance)} \\ & \sum_{(i,j)} m_{i,j} \leq \text{modulation} & \text{(off-peak demand-supply balance)} \\ & \sum_{t \in \mathcal{E}_{(i,j)}} x_{i,j,t} = 1 & \text{(assignement)} \\ & f_{i,j} \in [\underline{F}_{i,j}, \overline{F}_{i,j}] & \text{(final stock)} \\ & \sum_{\dots} x_{i,j,t} \leq \text{maxNumber} & \text{(maximal number parallel outages)} \\ & t_{(i,j)} - t_{(i',j')} \notin]N^l_{(i,j)}, \ N^l_{(i',j')}[& \text{(maximal lapping)} \end{array}\right.$$

1. (peak and off-peak demand-supply balance) — **2.** (assignement, final stock) — **3.** (maximal number parallel outages, maximal lapping)

1. demand-supply balance constraints ;
2. technical constraints ;
3. resources constraints.

Formulation of the maximal lapping constraint

Disjunctive constraints : $t_{(i,j)} - t_{(i',j')} \notin \left]N^l_{(i,j)},\ N^l_{(i',j')}\right[$: 3 possible formulations :

1. **Pairwise exclusion** : efficient but leads to a possibly very large number of constraints :

$$x_{i,j,t} + x_{i',j',t'} \leq 1 \text{ for all } t, t' \text{ such that } t - t' \in \left]-N^l_{i,j},\ N^l_{i',j'}\right[$$

2. **"Big M" formulation** : introduction of a new binary variable z, that equals 0 if $t_{i,j} - t_{i',j'} \leq N^l_{i,j}$ and 1 if $t_{i,j} - t_{i',j'} \geq N^l_{i',j'}$:

$$\begin{cases} t_{i,j} - t_{i',j'} \leq N^l_{i,j} + M_1 z \\ t_{i,j} - t_{i',j'} \geq N^l_{i',j'} - M_2(1-z) \\ z \in \{0,1\} \end{cases}$$

3. **Quadratic formulation** :

$$(t_{i,j} - t_{i',j'} - N^l_{i,j})(t_{i,j} - t_{i',j'} - N^l_{i',j'}) \geq 0$$

Approximation of the objective function by a quadratic function

Objective function :

- Nuclear cost : linear fonction of r and f ;
- Fossil-fuel cost : convex piecewise linear of p : one piece for each fossil-fuel kind of production (gas, coal, ...) ;

Piecewise linear :

$\Rightarrow$ one variable by piece and by time step.

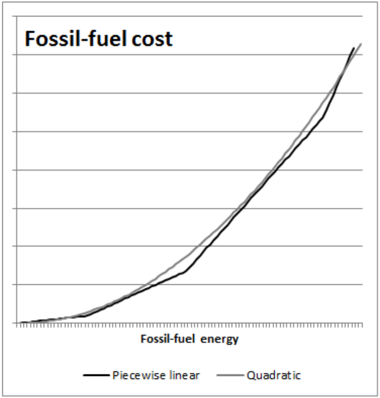
Approximation by a quadratic function :

$\Rightarrow$ one variable (p_t) by time step ;

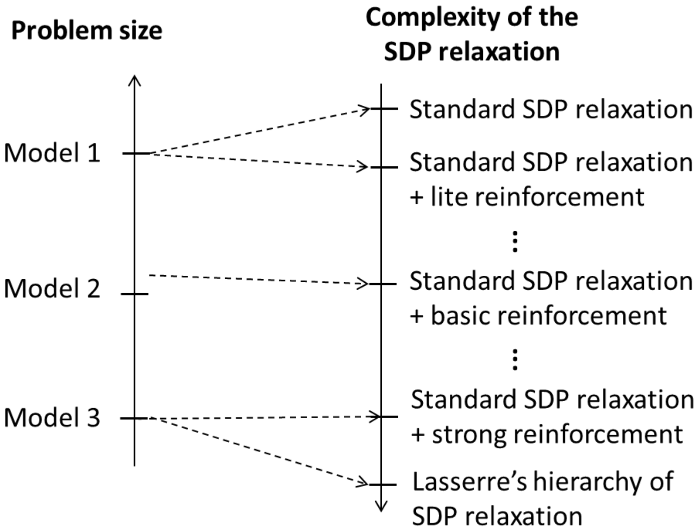
$\Rightarrow$ p_t can be replaced by its expression in function of x ;

$\Rightarrow$ no need to explicit the *peak demand* constraints ;

$\Rightarrow$ convex quadratic objective.



Outline



Problem size
Complexity of the SDP relaxation
Model 1
Model 2
Model 3
Standard SDP relaxation
Standard SDP relaxation + lite reinforcement
Standard SDP relaxation + basic reinforcement
Standard SDP relaxation + strong reinforcement
Lasserre's hierarchy of SDP relaxation

3 models

Model 1 $\rightarrow$ MIQP.
Purpose = test lite SDP relaxation :

- "big M" formulation of maximal lapping (all the constraints are linear);

Model 2 $\rightarrow$ Purely binary QCQP.
Purpose = test a systematic SDP relaxation :

- remove continuous variables (m, r, f);
- final stock constraint replaced by $\underline{T}_{(i,j)} \leq t_{i,j} - t_{i,j-1} \leq \bar{T}_{(i,j)}$;
- remove off-peak demand constraint;
- quadratic formulation of the maximal lapping constraint.

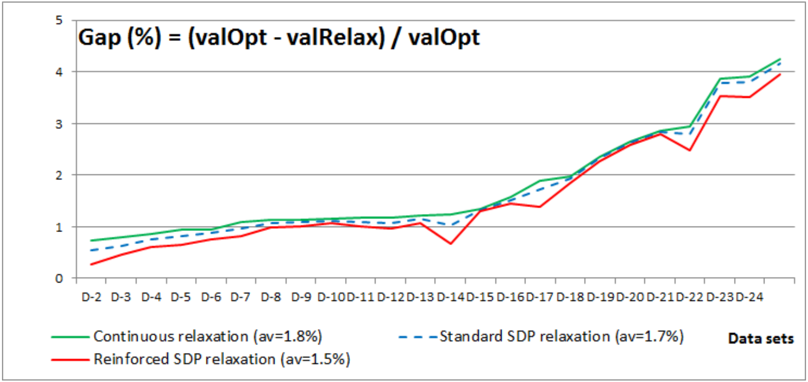
Model 3 $\rightarrow$ Purely binary QCQP (study model).
Purpose = compare various reinforcements proposed in the literature and test Lasserre's hierarchy :

- remove all the constraints but the assignement and maximal lapping ones;
- test the 3 formulations for the maximal lapping constraint.

Data sets

	Model 1	Model 2	Model 3
Structure	M- 0/1-QP	0/1-QCQP	0/1-QCQP
Data sets	3 years [10-20] plants 3 outages / plant [7-17] possible dates	[10-20] plants [3-4] outages / plant [7-15] possible dates	[2-8] plants 1 outage / plant [6-26] possible dates
Problem size	[200-1300] bin. var. [100-700] lin. csts 0 quad. csts	[200-1200] bin. var. [100-300] lin. csts [10-70] quad. csts	[10-200] bin. var. [20-2000] lin. csts [1-10] quad. csts

Model 1 : numerical results



Reinforcement : addition of valid quadratic equalities obtained by multiplying :

- assignment constraint : $\sum_{t \in \mathcal{E}_{(i,j)}} x_{i,j,t} = 1$
- $x_{i,j,t}$, for each each $x_{i,j,t}$ that appears in the assignement constraint.

$$x_{i,j,t'}(1 - \sum_{t \in \mathcal{E}_{(i,j)}} x_{i,j,t}) = 0, \ \forall t' \in \mathcal{E}_{i,j}$$

Model 2 : SDP relaxation

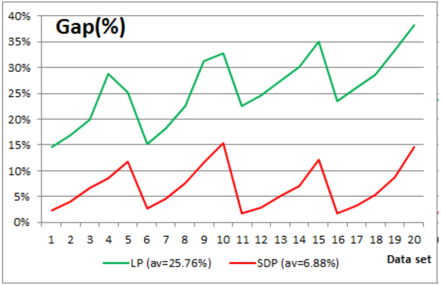
Design a separation algorithm to determinate the reinforcement :

1. Build a set P_V of valid constraints = **all the pairwise product of the linear constraints** of the problem (including bound constraints) ;
2. Select the most appropriate constraints of P_V = the most violated constraint by the current SDP solution ;
3. Add it to the SDP relaxation.

Comparison of 2 relaxations :

- *LP* : Linearization + continuous relaxation : resolution with CPLEX-LP ;

- *SDP* : SDP relaxations with systematic reinforcement (+ 100 constraints).



Model 3 : SDP relaxations

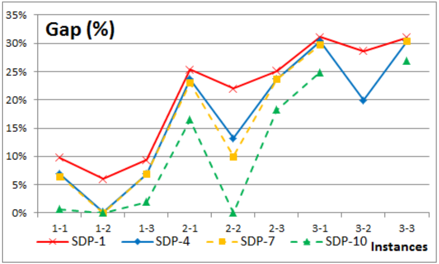
Comparison of 12 SDP relaxations proposed in the literature.

$\rightarrow$ 3 SDP relaxations selected :

- SDP-4 : linear constraints raised to the square ;
- SDP-7 : linear constraints multiplied by $x_i \geq 0$ and $1 - x_i \geq 0$;
- SDP-10 : SDP-7 + RLT (Reformulation-Linearization Technique) constraints ($x_i x_j \geq 0$, $x_i x_j \leq x_i$, $x_i x_j \leq x_j$ et $x_i x_j \geq 1 - x_i - x_j$) ;

9 instances = 3 data sets $\times$ 3 formulations of *maxLapping* constraints :

Data set	"BigM"	Pairwise	Quadratic
1	1-1	1-2	1-3
2	2-1	2-2	2-3
3	3-1	3-2	3-3

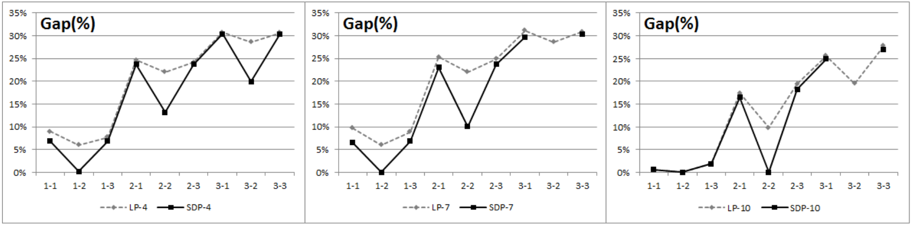


Problem 3-2 : no solution with SDP-10 due to its large size.

Model 3 : numerical results

Comparison with "reinforced" LP relaxation :

$$\text{QCQP} \xrightarrow{\text{Equiv.}} \text{QCQP}' \xrightarrow{\text{Relax.}} \begin{cases} \text{SDP} \\ \text{LP} \end{cases}$$



Conclusion :

- SDP-4 : good compromise between CPU and tightness ;
- SDP-7, SDP-10 : very tight, but difficult to apply ;
- SDP relaxation particularly appropriate on the model 2, because of a huge number of linear constraints.

Model 3 : application of the Lasserre's hierarchy

Lasserre's hierarchy = sequence of SDP relaxations of polynomial problems (Lasserre, 2008).

Principle : replace the monomials x^κ by a new variable y_κ and relax $y_\kappa = x^\kappa$ by semidefinite conditions over matrices containing y.

$$p_1^* \leq p_2^* ... \leq p_r^* \leq ... \leq p^* \quad \text{and} \quad \lim_{r \to +\infty} p_r^* = p^*$$

For 0/1-QCQP with n binary variables : $p_n^* = p^*$

Model 3 : application of the Lasserre's hierarchy (II)

Memory storage limits $\rightarrow$ applicable only for instances of size 1 and $r \leq 2$.

On average on the 100 instances :

Model	Gap		CPU (s)	
	Rank 1	Rank 2	Rank 1	Rank 2
"Big M"	9.74%	0.00%	0.01	2.42
Pairwise	6.03%	0.00%	0.01	1.06
Quadratic	9.39%	0.00%	0.02	4.72

$\rightarrow$ **reaches the optimal value at rank** 2 **on the** 100 **considered instances**

Size of the obtained SDP :

- Rank 1 : $n = [16 - 34]$ and $m = [55 - 66]$;
- Rank 2 : $n = [121 - 319]$ and $m = [385 - 561]$;

Conclusions

PhD objective : how to apply SDP to energy management ?

Our contributions :

- > selection of appropriate EMOPs and design of appropriate models ;
- > investigation of SDP relaxations for these problems :
  - – unification of different reinforcements proposed in the literature ;
  - – implementation of a systematic methods to identify efficient reinforcements.

SDP for adressing quadratic and combinatorial features of EMOPs ?

$\rightarrow$ provides convex relaxation of these problems, where the reinforcement can be seen as a mean of adjustment between accuracy and complexity.

Difficulties :

1. no consideration of CPU (large due to the infancy of the SDP solvers) ;
2. variability of the tightness of the obtained bound : identification of the problem's characteristics that lead to a tight bound ?

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