Reverse Chvátal-Gomory rank

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joint work with

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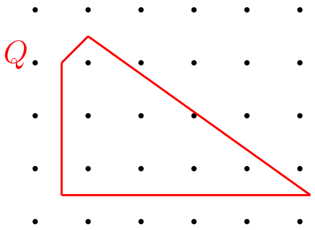
Y. Faenza, EPFL

Integer hull

Theorem (Meyer 1974)

The integer hull of a rational polyhedron is a polyhedron.



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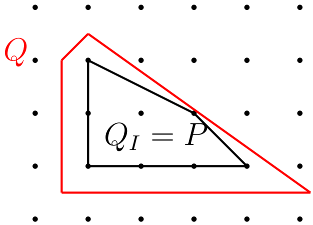

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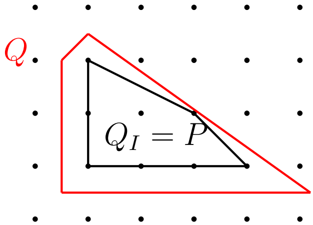

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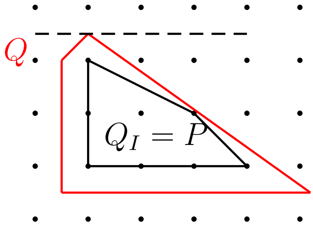

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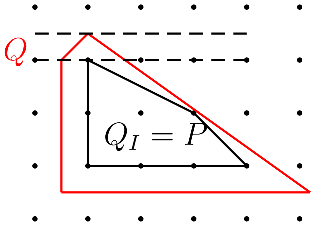

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Chvátal-Gomory closure

Chvátal-Gomory closure of Q: the set $Q^{(1)}$ obtained by applying CG rounding to all valid inequalities for Q.

Theorem (Schrijver 1980) If Q is a rational polyhedron, then $Q^{(1)}$ is a rational polyhedron.

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Theorem (Schrijver 1980) If Q is a rational polyhedron, then $Q^{(1)}$ is a rational polyhedron.

Iteratively define $Q^{(k)}$ as the Chvátal-Gomory closure of $Q^{(k-1)}$.

Theorem (Schrijver 1980) For every rational polyhedron Q, there is an integer k such that $Q^{(k)} = Q_I$.

Chvátal-Gomory rank

The smallest k such that $Q^{(k)} = Q_I$ is the Chvátal-Gomory rank of Q, denoted $r(Q)$.

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- If $Q \subseteq [0,1]^n$:
  - Bockmayr, Eisenbrand, Hartmann, Schulz 1999: $O(n^3 \log n)$; Eisenbrand, Schulz 1999/2003: $O(n^2 \log n)$;
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- $Q \subseteq [0,\ell]^n$
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Reverse Chvátal-Gomory rank

For an integral polyhedron P, we call a rational polyhedron Q such that $Q_I = P$ a relaxation of P.

For an integral polyhedron P, let

$$r^*(P) = \sup\{r(Q) : Q \text{ is a relaxation of } P\}.$$

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Questions

- When is $r^*(P) < +\infty$?
  - Recall: $r^*(P) < +\infty$ for $P = \emptyset$.
- When $r^*(P) < +\infty$, what does $r^*(P)$ depend on?

Main result

Theorem

Let $P \subseteq \mathbb{R}^n$ be a non-empty integral polytope.
Then $r^*(P) = +\infty$ if and only if there exists $v \in \mathbb{Z}^n \setminus \{0\}$ such that $P + \langle v \rangle$ is lattice-free.

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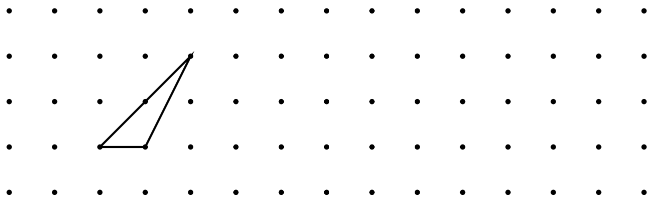

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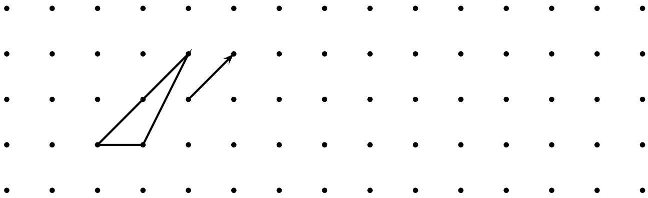

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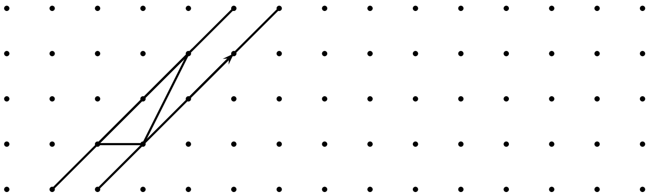

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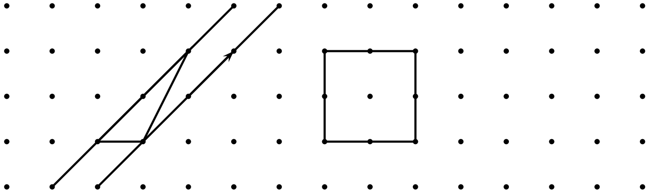

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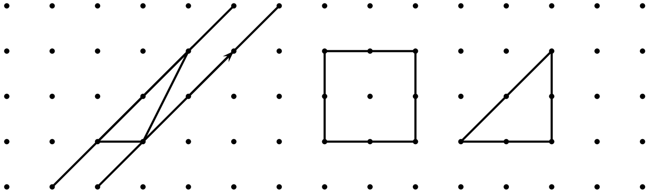

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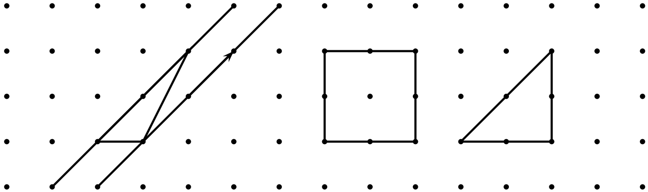
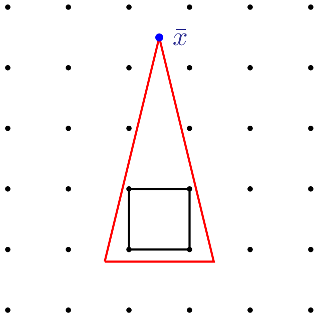
Let $P \subseteq \mathbb{R}^n$ be a non-empty integral polyhedron.
Then $r^*(P) = +\infty$ if and only if there exists $v \in \mathbb{Z}^n \setminus \mathrm{rec}(P)$ such that $P + \langle v \rangle$ is lattice-free.

- $\langle v \rangle$ is the 1-dimensional linear space generated by v;
- a set is lattice-free if it has no integer point in its relative interior;
- $\mathrm{rec}(P)$ is the recession cone of P, its set of unbounded directions.



Lower bound lemma

Let Q be a polyhedron and $\bar{x}$ be a point in Q.
Let v be an integer vector and $\bar{t} = \min\{t \geq 0 : \bar{x} + tv \in Q_I\}$.
Then $r(Q) \geq \lceil \bar{t} \rceil$.



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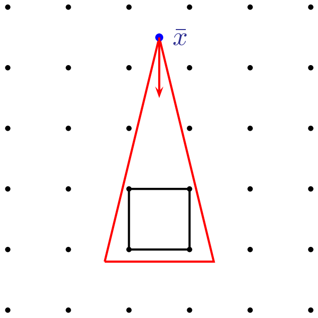

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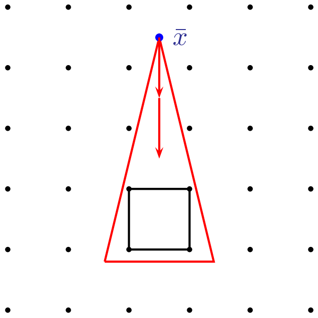

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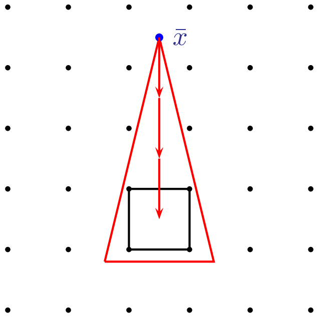

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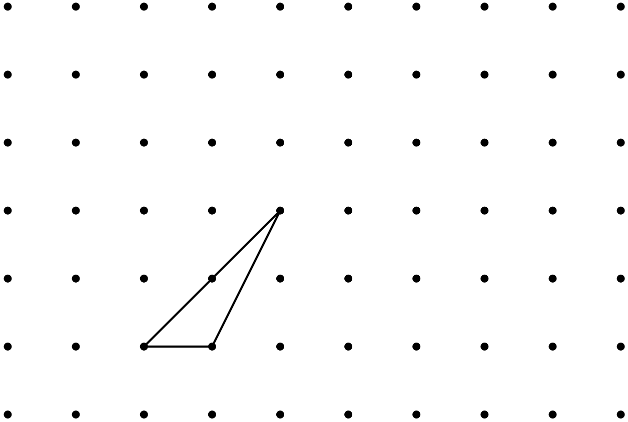

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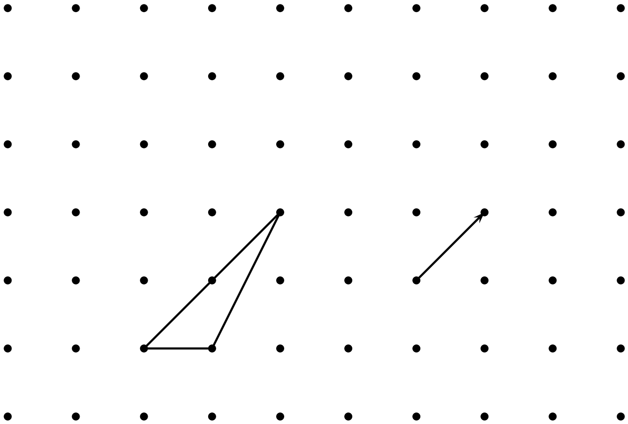

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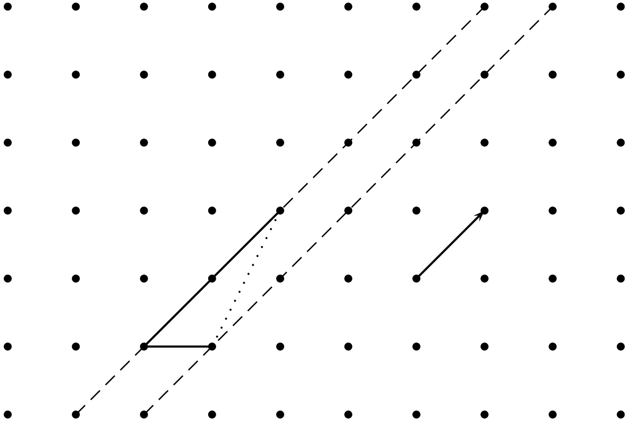

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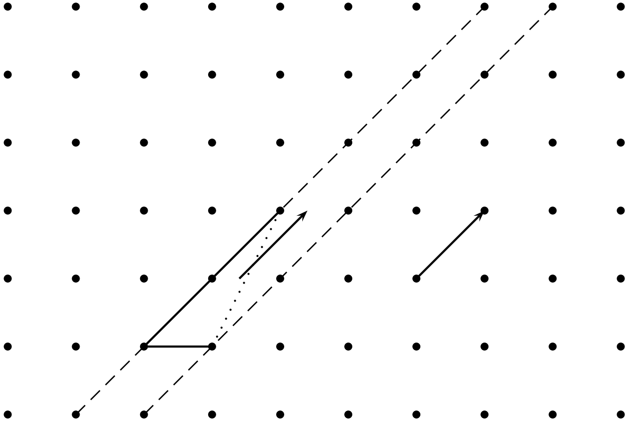

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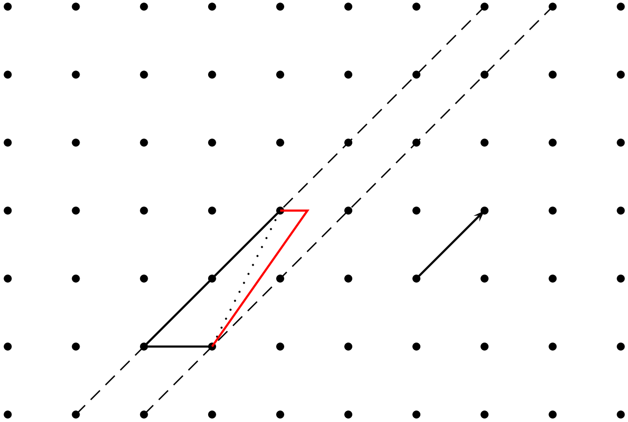

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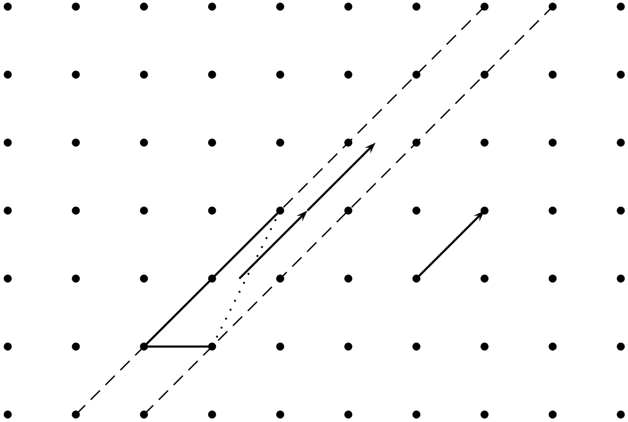

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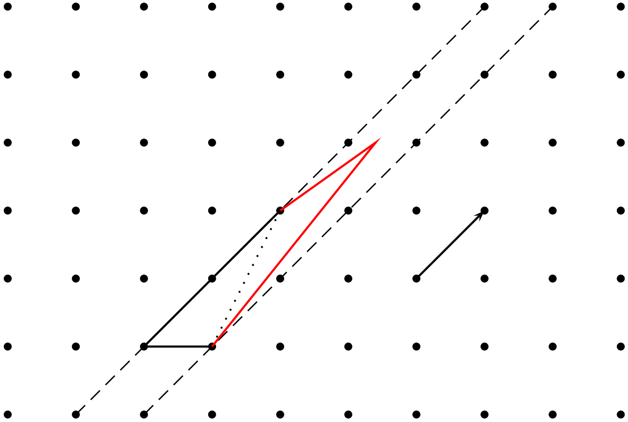

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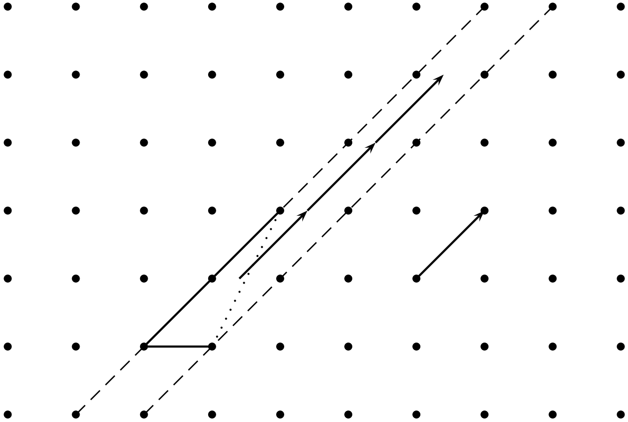

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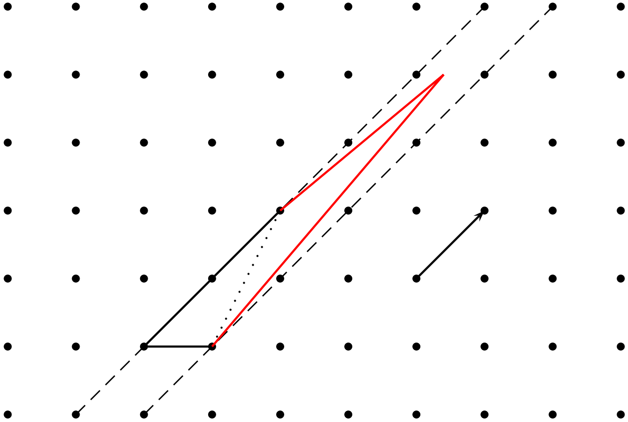

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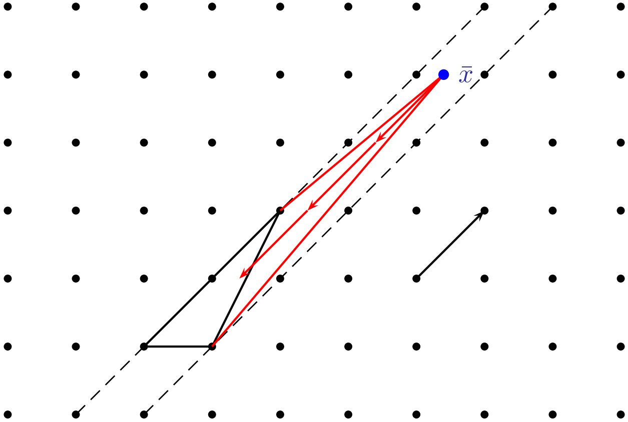

Upper-bound lemma

There exists a function $f(n)$, depending on n only, such that:
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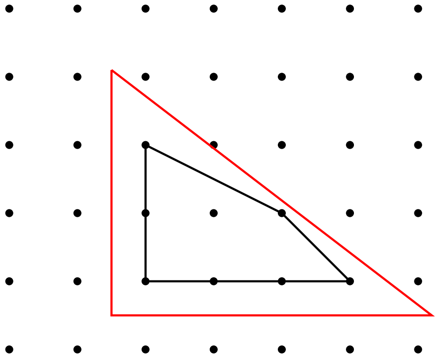

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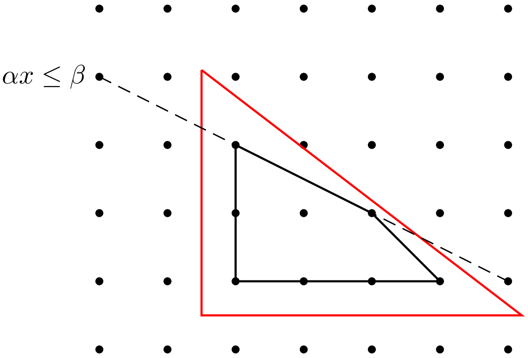

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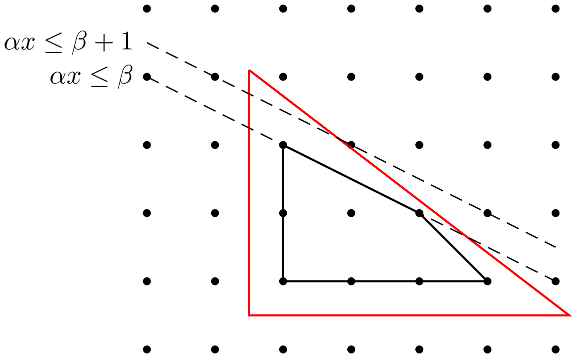

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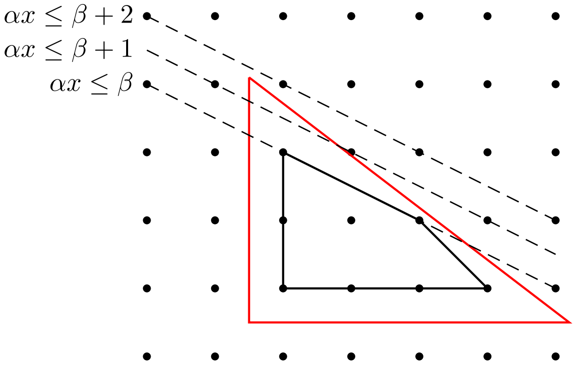

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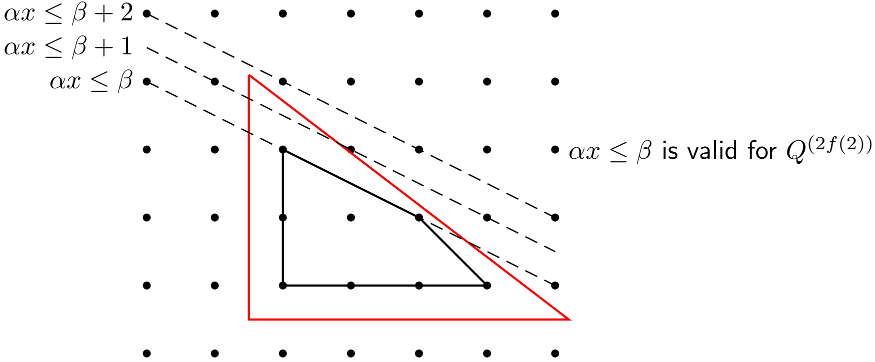

Necessity (for full-dimensional polytopes)

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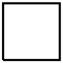

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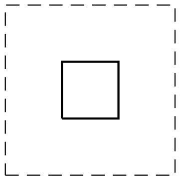

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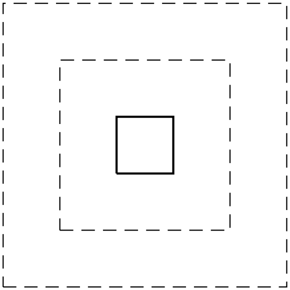

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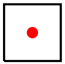

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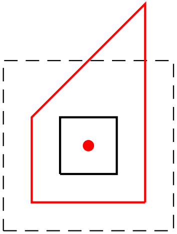

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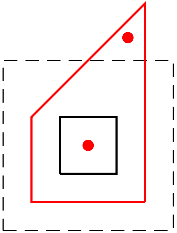

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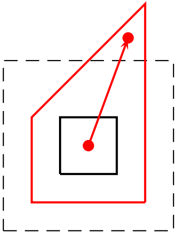

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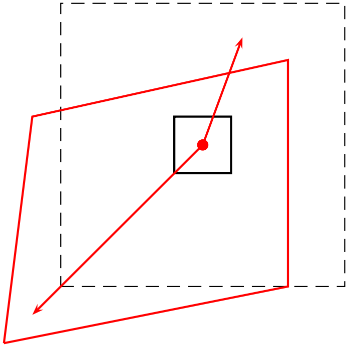

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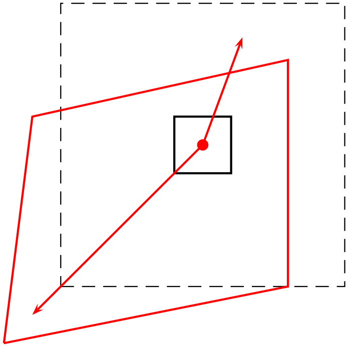

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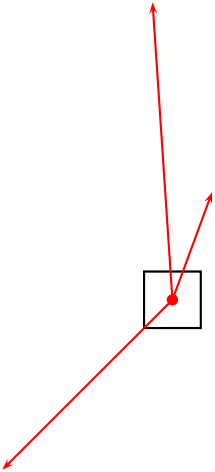

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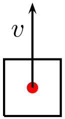

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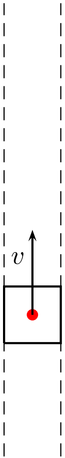

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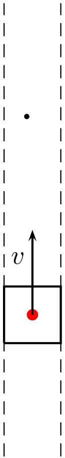

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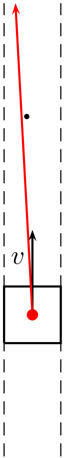

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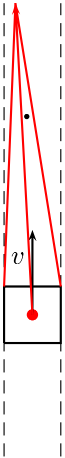

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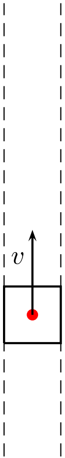

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1. construct v;
2. $P + \langle v \rangle$ is lattice-free;
3. we can assume $v \in \mathbb{Z}^n$

Upper bounds on $r^\star(P)$

Theorem

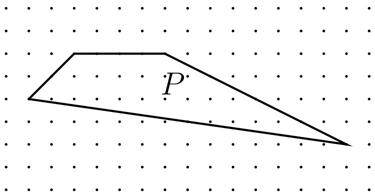
Let $\mathcal{A}_n$ be the family of full-dimensional polyhedron $P \subseteq \mathbb{R}^n$ such that none of the facets of P is lattice-free.
There exists a function $f(n)$ such that $r^*(P) \leq f(n)$ for every $P \in \mathcal{A}_n$.

- Idea: relaxations of P can not be "*too far*" from P.
- Upper-bound lemma applies.

Upper bounds on $r^*(P)$

$P \subseteq \mathbb{R}^2$, full-dimensional, every facet contains an integer point in its relative interior

Theorem $r^*(P) \leq f(2)$.



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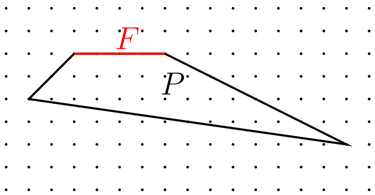

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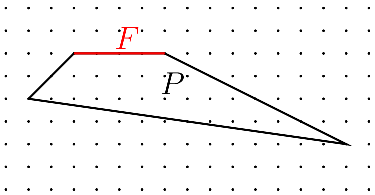
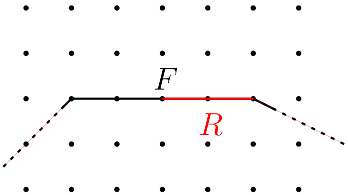


Zoom on F



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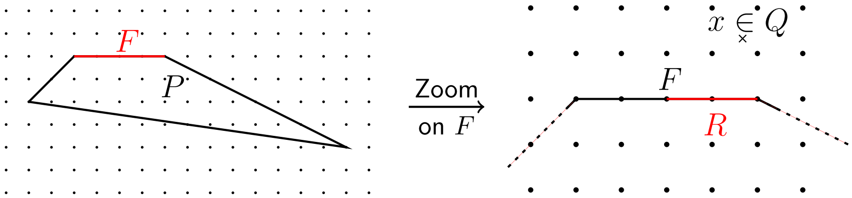

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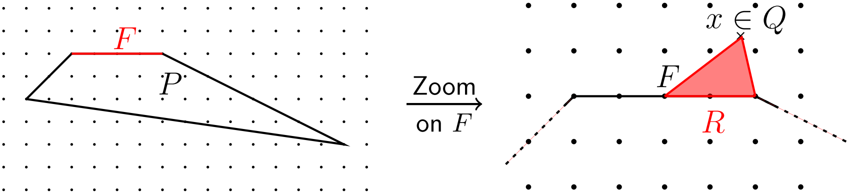

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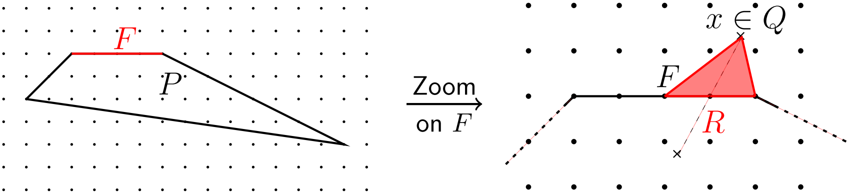

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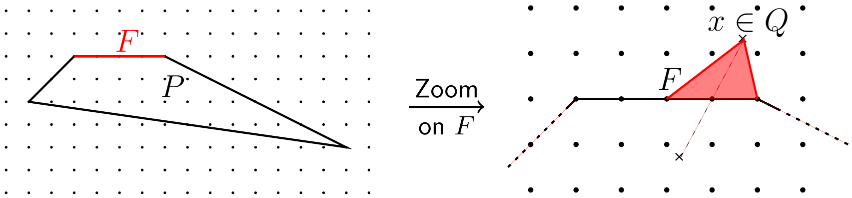

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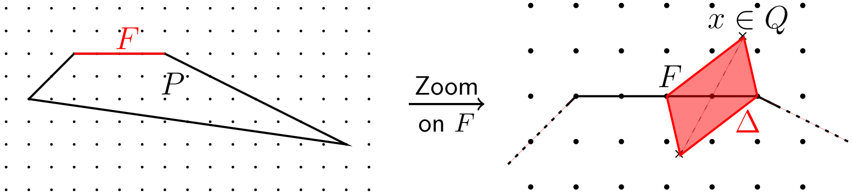

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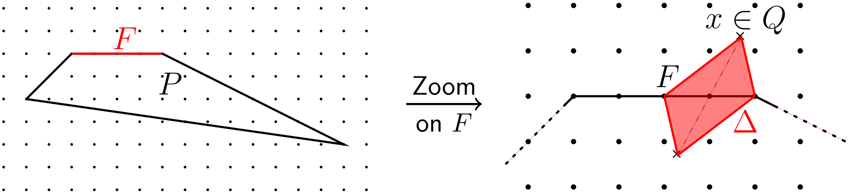

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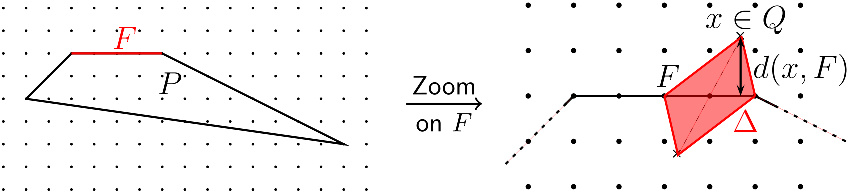

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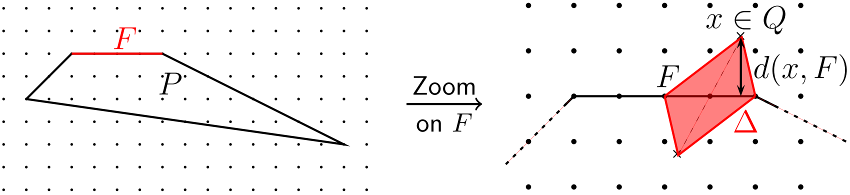

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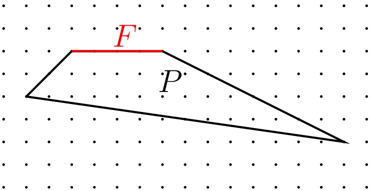

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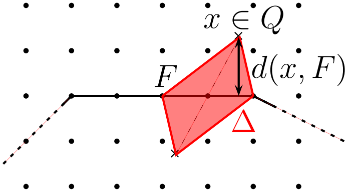

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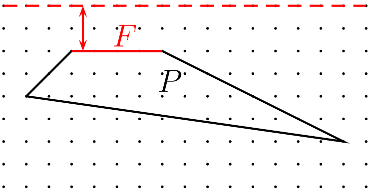

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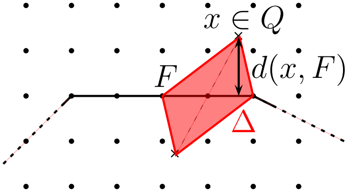

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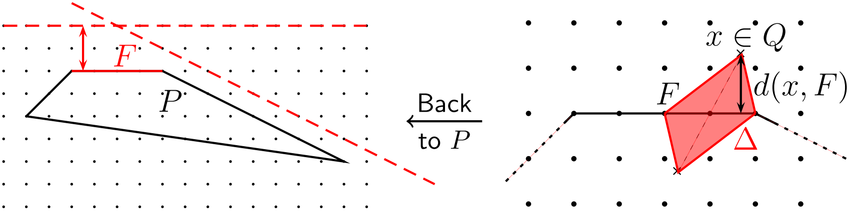

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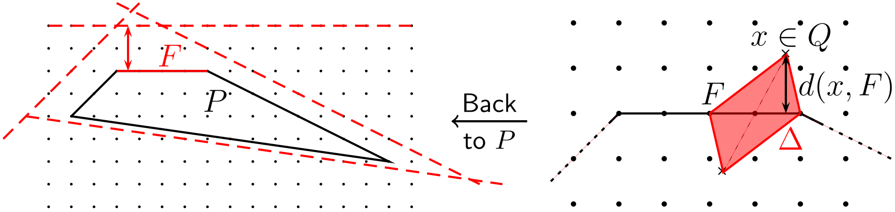

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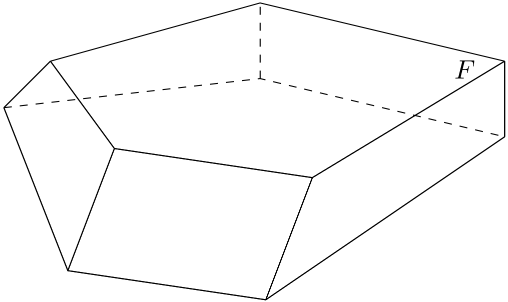

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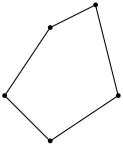

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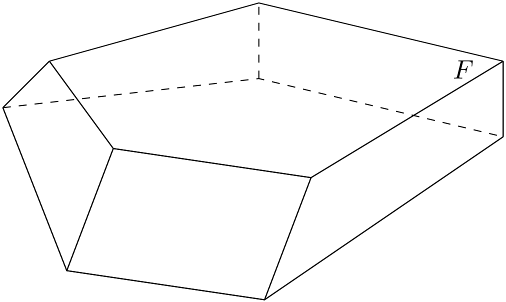

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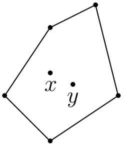

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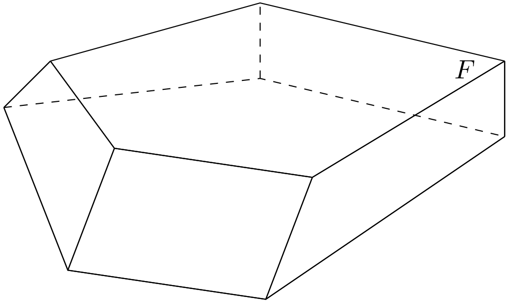

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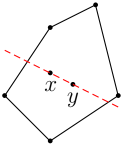

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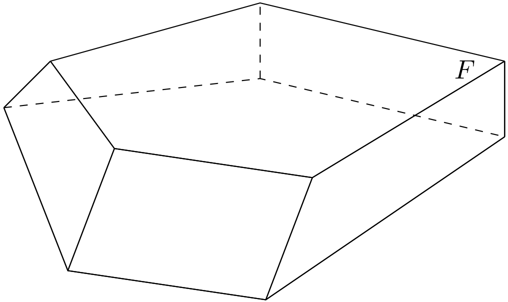

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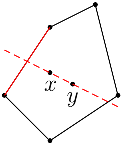

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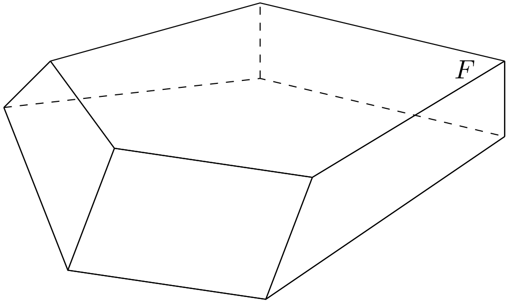

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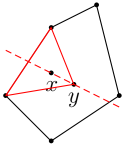

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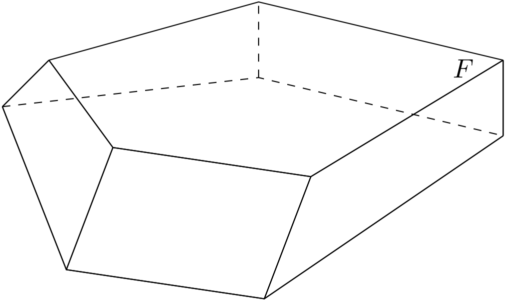

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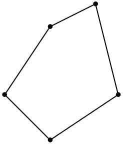

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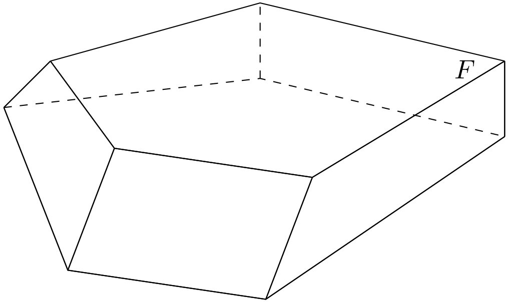

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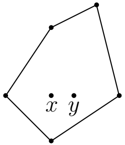

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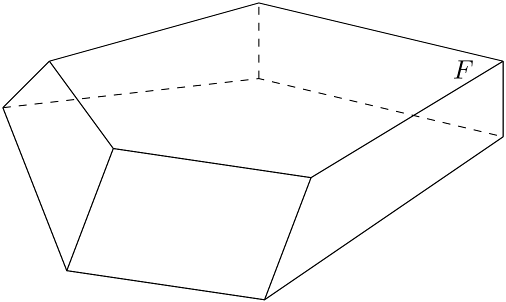

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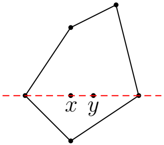

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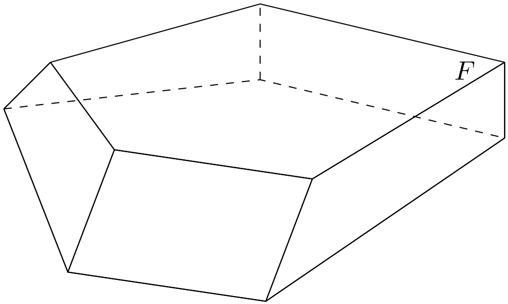

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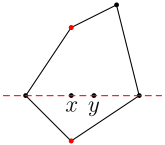

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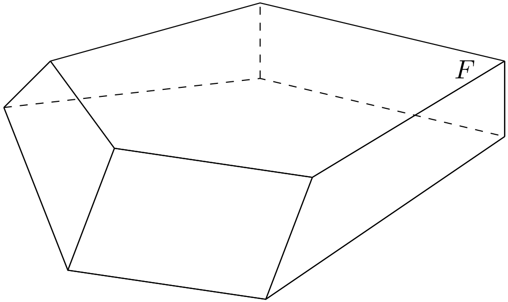

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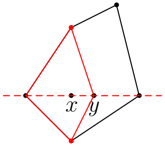

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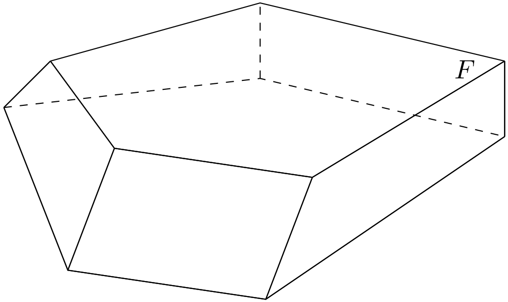

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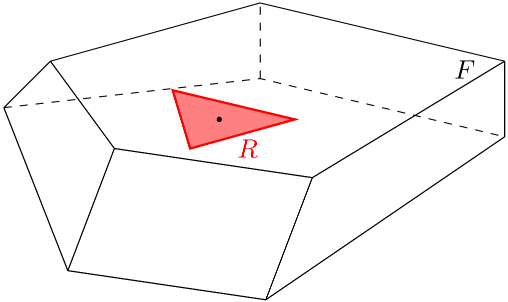

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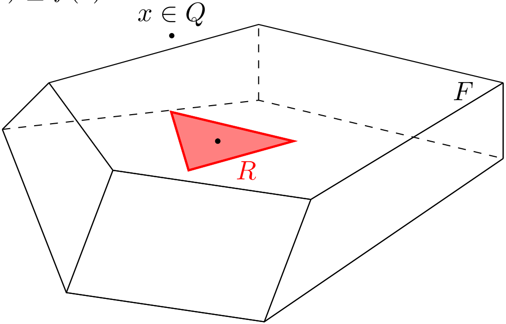

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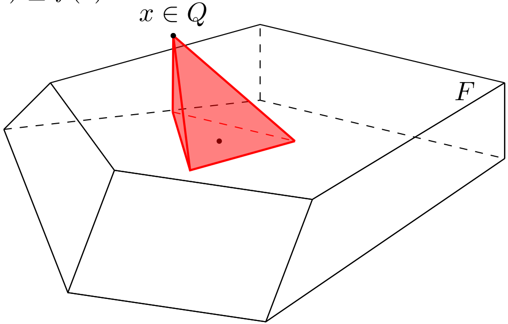

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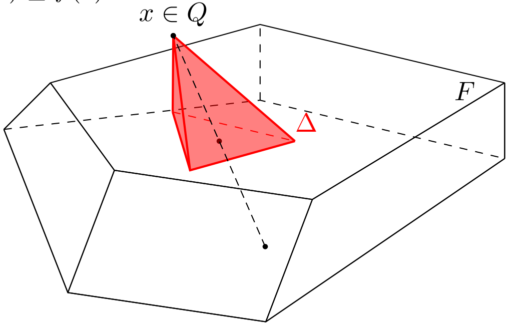

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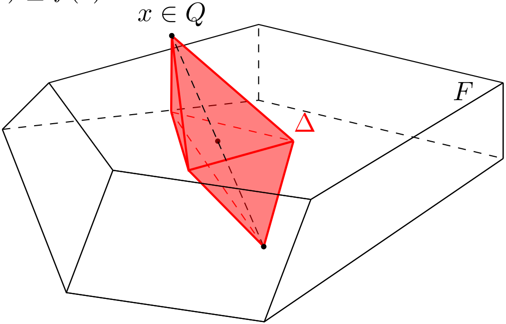

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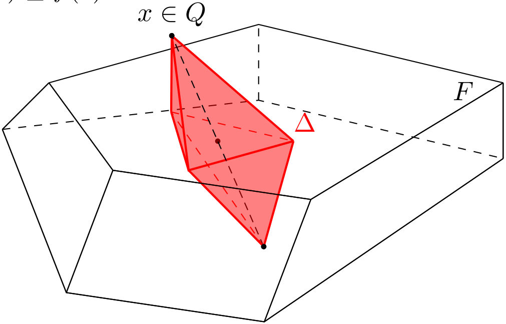

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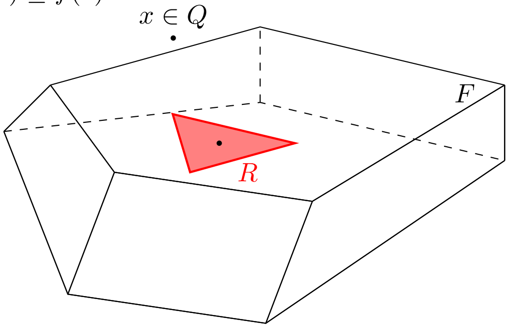

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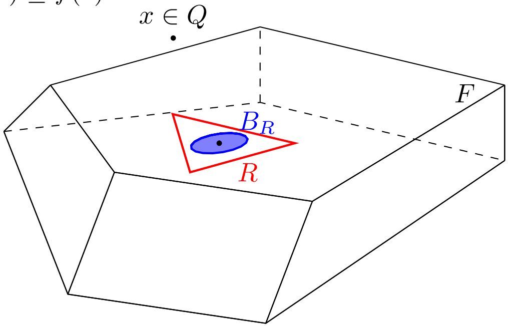

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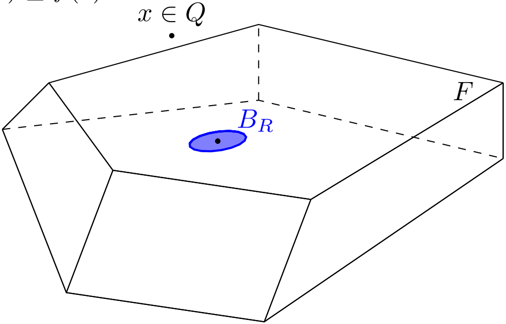

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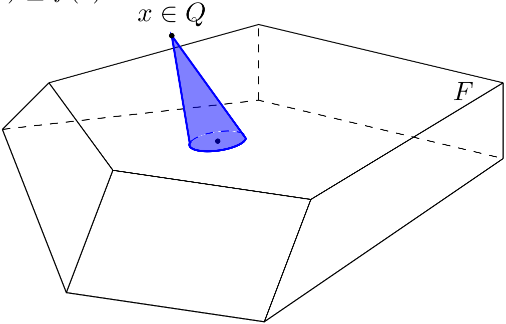

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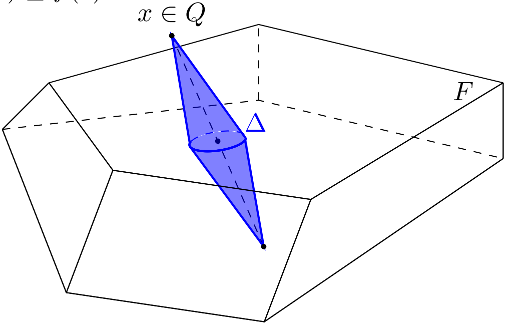

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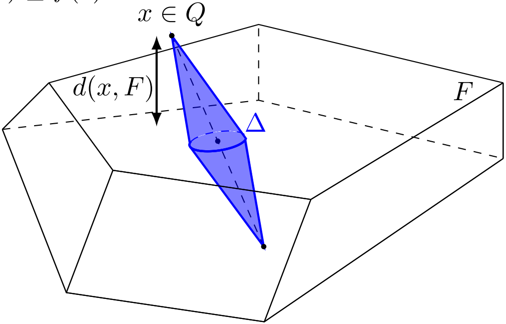

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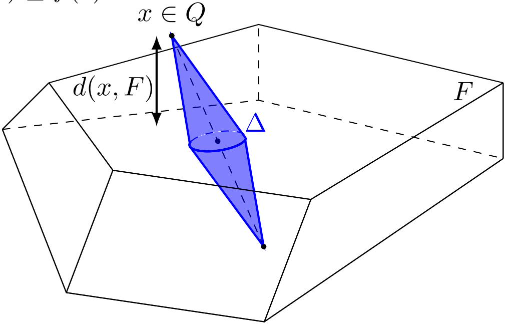

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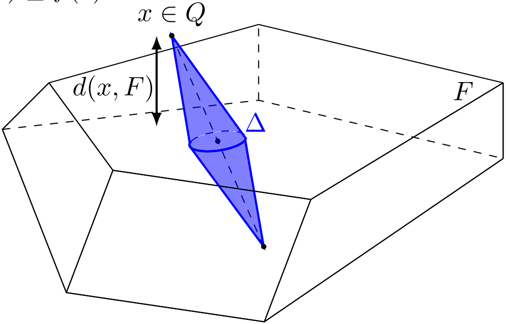

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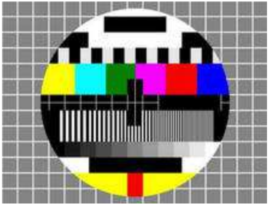

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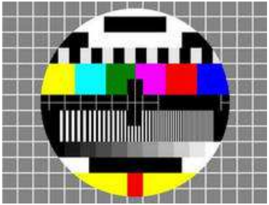

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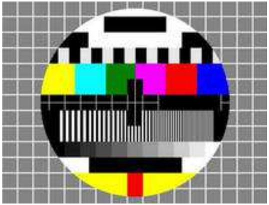

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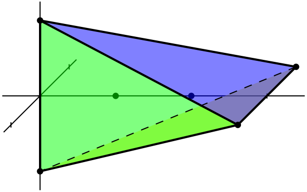

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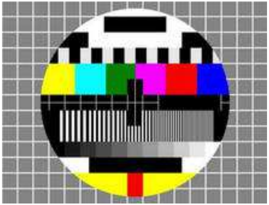

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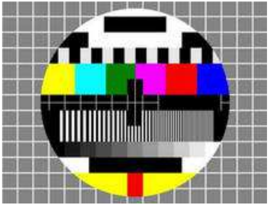

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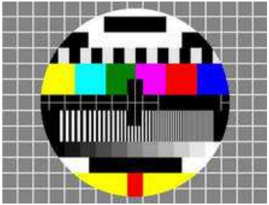

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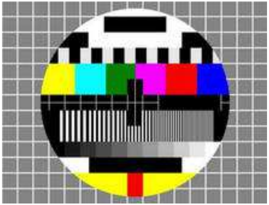

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Further results and future work

Bounds on r^*

- Let $\mathcal{A}_n$ be the family of polyhedra $P \subseteq \mathbb{R}^n$ such that none of the facets of P is lattice-free, and P is either full-dimensional or non lattice-free. There exists a function $f(n)$ such that $r^*(P) \leq f(n)$ for every $P \in \mathcal{A}_n$.

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- Let $\mathcal{B}_{n,k}$ be the family of non-lattice-free integral polyhedra $P \subseteq \mathbb{R}^n$ that contain k integer points in their interior. There exists a function $g(n,k)$ such that $r^*(P) \leq g(n,k)$ for every $P \in \mathcal{B}_{n,k}$.

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