On the use of copulae for the joint probabilistic constrained problems

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General chance constrained optimization problem

$$\min c(x) \quad \text{s.t.} \quad G(x) := \mathbb{P}\{g_k(x;\xi) \geq 0,\ k \in \mathcal{K}\} \geq p,\ x \in X_0 \tag{1}$$

$x \in \mathbb{R}^n$ ... decision vector
$X_0 \subset \mathbb{R}^n$ ... deterministic constraints
$\xi \in \mathbb{R}^s$ ... random vector
p ... fixed (high) probability
$\mathcal{K}$ ... index set for constraints g_k

Issues to address

- convexity of the feasible set
- computational effort.

$$\min c(x) \quad \text{s.t.} \quad G(x) = \mathbb{P}\{g_k(x;\xi) \geq 0,\ k \in \mathcal{K}\} \geq p,\ x \in X_0 \tag{1}$$

For simplicity, we assume

$$X_0 = \mathbb{R}^n \qquad c(x) = c^T x \qquad \mathcal{K} = \{1, \ldots, K\}.$$

Very classical result

Proposition 1

The feasible set is convex $\Leftrightarrow$ $G(x)$ is quasi-concave (on X_0).

We will concentrate in two important special cases

1. nonlinear constraints with random right-hand side
2. linear constraints with random coefficient matrix

Constraints with random right hand side

$$\min c^T x \quad \text{s.t.} \quad G(x) = \mathbb{P}\{g_k(x) \geq \xi_k,\ k \in \mathcal{K}\} \geq p \tag{2}$$

i. e.

$$g_k(x;\xi) = g_k(x) - \xi_k$$
$$G(x) = \mathbb{P}\{g_k(x) \geq \xi,\ k \in \mathcal{K}\} \geq p,$$

Denote

$$M(p) := \{x \mid \mathbb{P}\{g_k(x) \geq \xi_k,\ k \in \mathcal{K}\} \geq p\}$$

... set of feasible solutions of the problem (2).

Constraints with linear random matrix

$$\min c^T x \quad \text{s.t.} \quad G(x) = \mathbb{P}\{\Xi x \leq h\} \geq p \tag{3}$$

i. e.

$$\begin{aligned} \xi &= ({\Xi_1}^T, \ldots, {\Xi_k}^T) \\ g_k(x;\xi) &= h_k - \Xi x \\ G(x) &= \mathbb{P}\{\Xi_k x \leq h_k,\ k \in \mathcal{K}\} \geq p, \end{aligned}$$

Denote

$$X(p) := \{x \mid \mathbb{P}\{\Xi x \leq h\} \geq p\}$$

... set of feasible solutions of the problem (3).

Outline of the talk

1. Introduction — problem formulation
   - finished
2. Dependence of constraint rows
   - overcoming row independence conditions
   - notion of copulae
3. Convexity of the feasible set
   - classical results (first)
   - extended result
4. Approximation
   - formulating the problem as a second-order cone programming (SOCP) problem

Dependence

- Many practical results concerning chance-constrained programming is based on the assumption of **independence** of random rows of the problem.
- The proof are usually based on the separability property of the form

$$F(\xi) = \prod_{k \in \mathcal{K}} F_k(\xi_k)$$

  where F is the distribution function of ξ and F_k its marginals.
- first approach (Houda (2009)): approximate the problem employing the difference

$$F(\xi) - \prod_{k \in \mathcal{K}} F_k(\xi_k)$$

- this presentation: use of copulae (Nelsen (2006))

Definition 2

The copula is the distribution function $C : [0;1]^K \to [0;1]$ of some K-dimensional random vector whose marginals are uniformly distributed on $[0;1]$.

Proposition 3 (Sklar's theorem)

For any K-dimensional distribution function $F : \mathbb{R}^K \to [0;1]$ with marginals $F_1, \ldots, F_K$, there exists a copula C such that

$$\forall z \in \mathbb{R}^K \quad F(z) = C\big(F_1(z_1), \ldots, F_K(z_K)\big).$$

If, moreover, F_k are continuous, then C is uniquely given by

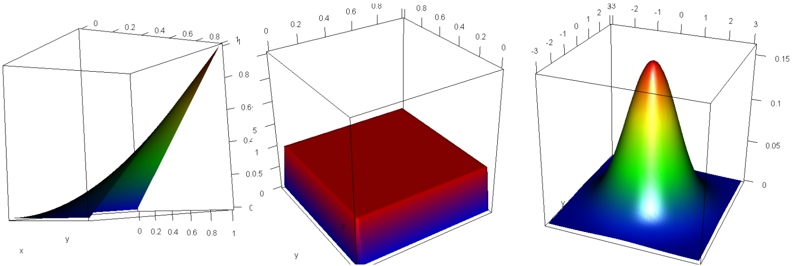
$$C(u) = F\left(F_1^{-1}(u_1), \ldots, F_K^{-1}(u_K)\right).$$

Otherwise, C is uniquely determined on $\operatorname{range} F_1 \times \cdots \times \operatorname{range} F_K$.

Assume $u = (u_1, \ldots, u_K)^T \in \mathbb{R}^K$

1. independence (product) copula

$$\Pi(u) := \prod_{k=1}^{K} u_k,$$



3. maximum (comonotone) copula

$$M(u) := \min_k \{u_k\}$$

We also define

$$W(u) := \max\left\{\sum_{k=1}^{K} u_k - K + 1, 0\right\}$$

(W fails to be a copula for $K > 2$)

Proposition 4 (Fréchet-Hoeffding bounds)

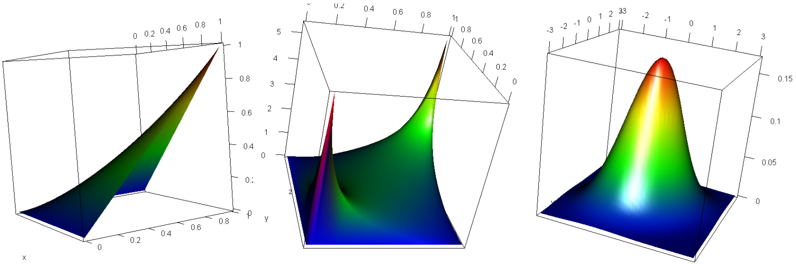
For every copula C and every $u \in \operatorname{dom} C$

$$W(u) \leq C(u) \leq M(u)$$

3. Gaussian (normal) copula

$$C^{\Sigma}(u) := \prod_{k} \Phi^{\Sigma}\left(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_K)\right)$$

where Φ^{Σ} is the distribution function of multivariate normal distribution with zero mean, unit variance and correlation matrix Σ



Definition 5

A copula C is called Archimedean if there exists a continuous strictly decreasing function $\psi : [0;1] \to \mathbb{R}_+$, called generator of C, such that $\psi(1) = 0$ and

$$C(u) = \psi^{-1}\left(\sum_{i=1}^{n} \psi(u_i)\right). \quad (4)$$

If $\lim_{u\to 0} \psi(u) = +\infty$ then C is called a strict Archimedean copula and ψ is called a strict generator.

Properties of copula generator

- ψ is convex
- ψ^{-1} is continuous strictly decreasing convex on $[0; \psi(0)]$

Proposition 6

Let $\psi : [0;1] \to \mathbb{R}^+$ *be convex, strictly decreasing function with* $\psi(1) = 0$, $\lim_{u\to 0} \psi(u) = +\infty$, *and*

$$(-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k} \psi^{-1}(t) \geq 0 \ \forall k = 0, 1, \ldots, K \text{ and } \forall t \in \mathbb{R}^+.$$

Then ψ *is a strict copula generator.*

Proposition 7

Let $\psi : [0;1] \to \mathbb{R}^+$ *be convex, strictly decreasing function with* $\psi(1) = 0$, $\lim_{u\to 0} \psi(u) = +\infty$. *Then* ψ *is a strict copula generator for every* $K \geq 2$ *iif*

$$(-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k} \psi^{-1}(t) \geq 0 \ \forall k = 0, 1, \ldots, \text{ and } \forall t \in \mathbb{R}^+.$$

Such function ψ^{-1} is called completely monotonic.

Practical advantages of Archimedean copulae:

- associative class (separability)
- explicit formulae of generators for most common copulae
- model dependence through one "nice" generator function and usually one or a small number of parameters
- many families adapted to a concrete problem setting (Nelsen (2006) provides a table of 22 one-parameter families of Archimedean copulae)

Some known shortcomings:

- Gaussian copula is not Archimedean
- all copula margins are the same $\Rightarrow$ only symmetric dependence structures described
- limited area of dependence structures (due to small number of parameters)
- less ability to capture negative dependence

1. independence (product) copula

$$\psi(t) := -\ln t,$$

2. Gumbel-Hougaard copulae, with $\theta \geq 1$

$$\psi_\theta(u) := (-\ln t)^\theta$$

Limiting cases

$$C_1 = \Pi \qquad C_{+\infty} = M$$

[see animation]

3. Clayton copulae, with with $\theta \geq 0$

$$\psi_\theta(u) := -\frac{1}{\theta}(t^{-\theta} - 1)$$

Limiting cases

$$C_0 = \Pi \qquad C_{+\infty} = M$$

[see animation]

$$\min c^T x \quad \text{s.t.} \quad G(x) = \mathbb{P}\{g_k(x) \geq \xi_k,\ k \in \mathcal{K}\} \geq p \tag{2}$$

Theorem 8

If the joint distribution of ξ is driven by an Archimedean copula with the generator ψ then the feasible set of chance-constrained problem (2) can be equivalently written as

$$M(p) = \Big\{ x \in X \mid \exists y_k \geq 0,\ \sum_{k=1}^{K} y_k = 1 : \\ \psi[F_k(g_k(x))] \leq \psi(p) y_k \text{ for every } k \in \mathcal{K} \Big\}. \tag{5}$$

where F_k are one-dimensional distribution functions of components ξ_k of ξ.

$$\min c^T x \quad \text{s.t.} \quad G(x) = \mathbb{P}\{\Xi x \leq h\} \geq p \tag{3}$$

- If the rows ${\Xi_k}^T$ are independent and follow $N_n(\mu_k; \Sigma_k \succ 0)$ then

$$\xi_k(x) := \frac{\Xi_k^T x - \mu_k^T x}{\sqrt{x^T \Sigma_k x}}, \qquad g_k(x) := \frac{h_k - \mu_k^T x}{\sqrt{x^T \Sigma_k x}},$$

  so

$$G(x) = \mathbb{P}\{g_k(x) \geq \xi_k(x),\ k \in \mathcal{K}\}$$

  where $\xi_k(x) \sim N(0; 1)$.

- If $K = 1$ (only one constraint) then

$$X(p) = \left\{ x \in X \mid {\mu_1}^T x + \Phi^{-1}(p)\sqrt{x^T \Sigma_1 x} \leq h_1 \right\}.$$

- To extend the result to $K > 1$ (more constraint rows): introduce auxiliary variables y_k. If the rows are independent, then (Cheng, Lisser(2012))

$$X(p) = \Bigg\{ x \in X \mid \exists y_k \geq 0,\ \sum_{k=1}^{K} y_k = 1 :$$

$$\qquad {\mu_k}^T x + \Phi^{-1}\left(p^{y_k}\right)\sqrt{x^T \Sigma_k x} \leq h_k \text{ for every } k \in \mathcal{K} \Bigg\}.$$

$$\min c^T x \quad \text{s.t.} \quad G(x) = \mathbb{P}\{\Xi x \leq h\} \geq p \tag{3}$$

Theorem 9

Suppose $\Xi_k^T \sim N(\mu_k, \Sigma_k \succ 0)$. *Then the feasible set of* (3) *is*

$$X(p) = \Big\{ x \in X \mid \exists y_k \geq 0,\ \sum_{k=1}^{K} y_k = 1 : \\ {\mu_k}^T x + \Phi^{-1}\left(\psi^{-1}(y_k \psi(p))\right) \sqrt{x^T \Sigma_k x} \leq h_k,\ k \in \mathcal{K} \Big\}$$

where

1. Φ *is one-dimensional standard normal distribution function, and*
2. ψ *is a generator of an Archimedean copula describing the dependence properties of the rows of the matrix* Ξ.

Definition 10

$f(\cdot)$ is r-concave iif for any x, y such that $f(x) > 0, f(y) > 0$ and $\lambda \in [0; 1]$ we have

$$f(\lambda x + (1-\lambda)y) \geq [\lambda f^r(x) + (1-\lambda) f^r(y)]^{1/r}$$

cases $r = -\infty, 0, +\infty$ treated by continuity.

$r = -\infty$	$RHS = \min\{f(x), f(y)\}$	... f quasi-concave
$r < 0$		... f^r convex
$r = 0$	$RHS = f^\lambda(x) f^{1-\lambda}(y)$	... f log-concave ($\log f$ concave)
$r > 0$		... f^r concave
$r = 1$		... f concave
$r = +\infty$	$RHS = \max\{f(x), f(y)\}$	... f quasi-convex

If f is r^*-concave, it is also r-concave for all $r \leq r^*$
$\Rightarrow$ every r-concave function is quasi-concave.

Prékopa (1971) (log-concave measures)

Definition 11

Probability measure $\mathbb{P}$ (on $\mathcal{B}(\mathbb{R}^s)$) is r-concave iif for any Borel convex subset A, B such that $\mathbb{P}(A) > 0, \mathbb{P}(B) > 0$ and $\lambda \in [0;1]$ we have

$$\mathbb{P}(\lambda A + (1-\lambda)B) \geq [\lambda \mathbb{P}^r(A) + (1-\lambda)\mathbb{P}^r(B)]^{1/r}$$

cases $r = -\infty, 0, +\infty$ treated by continuity.

Properties:

1. r-concave probability measure has r-concave distribution function
2. quasi-concave measure $\mathbb{P}$ on $\mathbb{R}^s$ with $\dim \operatorname{supp} \mathbb{P} = s$ has a density
3. r-concave density $\Leftrightarrow$ $\frac{r}{1+mr}$-concave measure on convex subset $\Omega \subset \mathbb{R}^s$ of dim $m > 0$ (so for $r > -\frac{1}{m}$)

Borell (1975), Brascamp, Lieb (1976)

Theorem 12

If

1. $g_k(\cdot,\cdot), k \in \mathcal{K}$ ... *quasi-concave jointly in both arguments*
2. $\xi \in \mathbb{R}^s$ ... *random variable with r-concave probability distribution*

then the feasible set

$$X_p := \{x \mid G(x) \geq p\}$$

is convex and closed.

- consequence of a general theorem by Prékopa (1971), see also Prékopa (1995) Shapiro, Ruszczyński, Dentcheva (2009))
- in fact, (ii) means quasi-concave probability distribution (or $-\frac{1}{s}$-concave density);
- many prominent multivariate distributions has log-concave (or at least quasi-concave and/or for some parameters) distribution so that (ii) is not tie (uniform, normal, Wishart, Beta, Dirichlet, Gamma, Cauchy, Pareto)

$$\min c^T x \quad \text{s.t.} \quad G(x) = \mathbb{P}\{g_k(x) \geq \xi_k,\ k \in \mathcal{K}\} \geq p \qquad (2)$$

Recall $g_k(x;\xi) = g_k(x) - \xi_k$

- $g_k(x;\xi)$ required quasi-concave in both arguments
- Issue: quasi-concavity not preserved under additions
- Classical result: if $g_k(\cdot)$ concave $\Rightarrow$ $g_k(\cdot;\cdot)$ concave $\Rightarrow$ assumption (i) of Theorem 12 fulfilled.

Idea: Henrion, Strugarek (2008)

- relax concavity requirement on $g_k(\cdot)$
- strengthen quasi-concavity requirement on distribution of ξ

Definition 13 (Henrion, Strugarek (2008))

$f_\xi(\cdot)$ is an r-decreasing density of one-dimensional random variable ξ if

1. f_ξ is continuous on $(0; +\infty)$
2. $t^r f_\xi(t)$ is (strictly) decreasing for all $t > t^*$ and some treshold $t^* > 0$

- $r = 0$ ... density decreasing in classical sense
- f_ξ r^*-decreasing for some $r^* \Rightarrow r$-decreasing for all $r \geq r^*$

Lemma 14

The normal distribution has r-decreasing density for every $r > 0$ with threshold $t^* = \frac{1}{2}\left(\mu + \sqrt{\mu^2 + 4r\sigma^2}\right)$

- for standard normal distribution: $t^* = \sqrt{r}$

$$\min c^T x \quad \text{s.t.} \quad G(x) = \mathbb{P}\{g_k(x) \geq \xi_k,\ k \in \mathcal{K}\} \geq p \tag{2}$$

Proposition 15 (Henrion, Strugarek (2008))

If

1. $g_k(\cdot)$ *are* $(-r_k)$*-concave with* $r_k > 0$
2. ξ_k *are independent with* $(r_k + 1)$*-decreasing densities with thresholds* $t_k^* > 0$
3. $p > \max\{F_k(t_k^*)\}$

Then the feasible set $M(p)$ *of* (2) *is convex.*

- Henrion and Strugarek(2011) generalized the result for dependency given by so-called *log-exp-concave* copulae
- our result similar, but based on properties of Archimedean copulae

$$\min c^T x \quad \text{s.t.} \quad G(x) = \mathbb{P}\{g_k(x) \geq \xi_k,\ k \in \mathcal{K}\} \geq p \tag{2}$$

Theorem 16

If

1. $g_k(\cdot)$ *are* $(-r_k)$*-concave with* $r_k > 0$
2. *the marginal distributions functions* F_k *have* $(r_k + 1)$*-decreasing densities with thresholds* t_k^*
3. *the joint distribution of* ξ *is driven by an Archimedean copula with a strict generator* ψ *such that* ψ^{-1} *is completely monotonic;*
4. $p > \max\{F_k(t_k^*)\}$

Then the feasible set $M(p)$ *of* (2) *is convex.*

- independent, Gumbel-Hougaard, Clayton (and many other) copulae satisfy assumption (iii) of Theorem 16

Convexity of the feasible set: main result

Chance-constrained problem with dependent constraint rows

$$\min c^T x \quad \text{s.t.} \quad G(x) = \mathbb{P}\{\Xi x \leq h\} \geq p \tag{3}$$

Theorem 17

If

1. *the rows* $\Xi_k^T \sim N(\mu_k, \Sigma_k \succ 0)$*;*
2. *the joint distribution function of standardized rows is driven by an Archimedean copula with a generator* ψ

then the problem (3) *can be equivalently written as*

$$\min c^T x \quad s.t. \quad {\mu_k}^T x + \Phi^{-1}\left(\psi^{-1}(y_k \psi(p))\right)\sqrt{x^T \Sigma_k x} \leq h_k,$$
$$\sum\nolimits_k y_k = 1, \; y_k \geq 0, \; k \in \mathcal{K}.$$

Moreover, if

3. ψ^{-1} *is completely monotonic;*
4. $p > p^* := \Phi\left(\max\{\sqrt{3}, 4\lambda_{max}^{(k)}[\lambda_{min}^{(k)}]^{-3/2}\|\mu_k\|\}\right)$*, where* $\lambda_{max}^{(k)}, \lambda_{min}^{(k)}$ *are largest and lowest eigenvalues of the matrices* Σ_k*,*

then the problem is convex.

Lemma 18

The function $y \mapsto \Phi^{-1}\left(\psi^{-1}(y_k \psi(p))\right)$ *is convex.*

Consequence: linear approximation results in

1. piecewise tangent approximation ($\Rightarrow$ outer bound for feasible solutions)
2. piecewise linear approximation ($\Rightarrow$ inner bound for feasible solutions)

Hence we obtain two SOCP problems of the form

$$\min c^T x \quad \text{s.t.} \quad \begin{aligned} & {\mu_k}^T x + \sqrt{{z^k}^T \Sigma_k z^k} \le h_k, \\ & z^k \ge a_{kj} x + b_{kj} w^k, \ k \in \mathcal{K}, \ j \in \mathcal{J} \\ & \textstyle\sum_k w^k = x, \ w^k \ge 0, \ z^k \ge 0, \ k \in \mathcal{K}. \end{aligned}$$

$\mathcal{J}$ is the index set of partition points of $[0;1]$, a_{kj} and b_{kj} are given by the linear approximation above.

Details not presented here (technique described in Cheng, Lisser(2012)).

Conclusions

In this presentation

- Equivalent deterministic formulation of the linear chance-constrained problems with random matrix of dependent rows was given.
- Row dependence modeled through Archimedean copulae.
- Convexity of the feasible set proven for sufficiently high probabilities.
- Inner and outer approximation of the feasible set formulated as SOCP problems.

Under actual investigation

- possible extensions over normal distribution
- checking different class of Archimedean copulae for the convexity
- copula estimation
- numerical tests (comparison with other numerical methods)
- application in power industry

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