A completely positive representation of 0-1 Linear Programs with Joint Probabilistic Constraints

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Outlines

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- We study the following $0-1$ linear program with joint probabilistic or chance constraints:

LPJPC

$$
\begin{array}{lll}
 & \max & c^T x \\
(LPJPC) & s.t. & \mathbf{Pr}\{Tx \le D\} \ge 1-\alpha \quad (1)\\
 & & w_t^T x = d_t, \quad t = 1,\ldots,m \quad (2)\\
 & & \bar{w}_{\bar{t}}^T x \le \bar{d}_{\bar{t}}, \quad \bar{t} = 1,\ldots,\bar{m} \quad (3)\\
 & & x \in \{0,1\}^n
\end{array}
$$

where $c \in R^n$, $w_t \in R^n$, $\bar{w}_{\bar{t}} \in R^n$, $D = (D_1,\ldots,D_K) \in R^K$,

- $T = [T_1,\ldots,T_K]^T$ is a $K \times n$ random matrix, where $T_k, k = 1,\ldots,K$ is a random vector in R^n, and α is a prespecified confidence parameter, in general assuming that $\alpha \le \frac{1}{2}$.
- When $K \ge 1$, the chance constraint in (1) specifies that all constraints are to be jointly satisfied with a given probability.

- We assume that $T_k, k = 1, \ldots, K$ are independant multivariate normally distributed with known mean vector $\mu_k = (\mu_{k1}, \ldots, \mu_{kn})$ and covariance matrix Σ_k.
- We can derive a deterministic reformulation of normally distributed LPJPC as follows:

DEP

$$
\begin{aligned}
\max \quad & c^T x \\
s.t. \quad & \mu_k^T x + F^{-1}(p^{z_k}) ||\Sigma_k^{1/2} x|| \leq D_k, k = 1, ..., K \\
(P) \quad & \sum_{k=1}^{K} z_k = 1, \quad z_k \geq 0, k = 1, ..., K \\
& w_t^T x = d_t, \quad t = 1, \ldots, m \\
& \bar{w}_{\bar{t}}^T x \leq \bar{d}_{\bar{t}}, \quad \bar{t} = 1, \ldots, \bar{m} \\
& x \in \{0, 1\}^n
\end{aligned}
\tag{4}
$$

- where $p = 1 - \alpha$ and $F^{-1}(\cdot)$ is the inverse of F which is the standard normal cumulative distribution function.

DEP

- The problem (4) is equivalent to

$$
\begin{array}{ll}
\max & c^T x \\
s.t. & (F^{-1}(p^{z_k}))^2 x^T \Sigma_k x \leq (D_k - \mu_k^T x)^2, k = 1, ..., K \\
 & \mu_k^T x \leq D_k, k = 1, ..., K \\
(P) & \sum_{k=1}^{K} z_k = 1, \quad z_k \geq 0, k = 1, ..., K \\
 & w_t^T x = d_t, \quad t = 1, \ldots, m \\
 & \bar{w}_{\bar{t}}^T x \leq \bar{d}_{\bar{t}}, \quad \bar{t} = 1, \ldots, \bar{m} \\
 & x \in \{0, 1\}^n
\end{array}
$$

- There are two steps to approximate the problem (4):
  1. Firstly, we approximate $(F^{-1}(p^{z_k}))^2$ with a piecewise tangent approximation of z_k and the approximations of NLPPC is an SOCP problem apart from the binary constraints.
  2. Secondly, we transform the SOCP constraints into linear constraints by using linearization method.

Lemma

With the piecewise tangent approximation of $(F^{-1}(p^{z_k}))^2$ and linearization method, we have the approximation of (P) as follows:

$$
\begin{aligned}
OPT_{AP} = \max \quad & c^T x \\
s.t. \quad & \langle \Sigma_k, Z^k \rangle \leq D_k^2 - 2D_k \mu_k^T x + \langle \mu_k \mu_k^T, X \rangle, \quad k = 1, \ldots, K \\
& \mu_k^T x \leq D_k, \quad \sum_{k=1}^{K} z_k = 1, \quad z_k \geq 0, k = 1, \ldots, K \\
& Z_{i,j}^k \leq U^+ X_{ij}, \quad Z_{i,j}^k \leq \hat{F}_k, \quad Z_{i,j}^k \geq 0 \\
& Z_{i,j}^k \geq \hat{F}_k - (1 - X_{ij})U^+, \quad i, j = 1, \ldots, n, \quad k = 1, \ldots, K \\
& \hat{F}_k \geq a_l + b_l z_k, \quad l = 0, 1, \ldots, N, \\
(AP) \quad & X_{ij} \leq x_i, \quad X_{ij} \leq x_j, \quad X_{ij} \geq 0 \\
& X_{ij} \geq x_i + x_j - 1, \quad i, j = 1, \ldots, n \\
& X_{ii} = x_i, \quad i = 1, \ldots, n \\
& w_t^T x = d_t, \quad t = 1, \ldots, m \\
& \bar{w}_{\bar{t}}^T x \leq \bar{d}_{\bar{t}}, \quad \bar{t} = 1, \ldots, \bar{m} \\
& x \in \{0, 1\}^n
\end{aligned}
\tag{5}
$$

- *where $a_0 = 0$, $b_0 = 0$, U^+ is an upper bound of $\hat{F}_k$ and $\hat{F}_k = \max_{l=0,\ldots,N}\{a_l + b_l \cdot z_k\}$ is a piecewise tangent approximation of $(F^{-1}(p^{z_k}))^2$. Moreover, the optimal value of (AP) is an upper bound of (P). Furthermore, $\lim_{N \to \inf} OPT_{AP} = OPT_P$.*

Proof.

- Firstly, we prove that $(F^{-1}(p^z))^2$ is convex on the interval $(0,1]$.
- The second derivative of $F^{-1}(p^z)$ is given by:

$$(F^{-1}(p^z))'' = \frac{(\ln p)^2 p^z [f(F^{-1}(p^z)) + F^{-1}(p^z)]}{[f(F^{-1}(p^z))]^2}$$

- As $p \geq \frac{1}{2}$, then $F^{-1}(p^z)$ is nonnegative. Thus $(F^{-1}(p^z))''$ is nonnegative and $F^{-1}(p^z)$ is convex.
- Moreover x^2 is non-decreasing and convex on the interval $[0,\infty)$. Therefore, $(F^{-1}(p^z))^2$ is convex.
- In the second step, by appling the standard linearization technique introduced in Fortet(1959) and the theory presented in Cheng and L. (2012), it shows the optimal value of (AP) is an upper bound of (P).
- Finally, one applies the results of the piecewise linear approximation presented in Thakur (1978).

□

- By adding the slack variables $s_k, \hat{s}_k, \hat{s}_{kl}, \bar{s}_{\bar{t}}, Z^{k'}, Z^{k''}, Z^{k'''}, X'_{ij}, X''_{ij}, X'''_{ij}$, we get the standard formulation as in Burer (2009):

OPT-AP

$$
\begin{aligned}
OPT_{AP} = \max \quad & c^T x \\
s.t. \quad & \langle \Sigma_k, Z^k \rangle + 2D_k \mu_k^T x - \langle \mu_k \mu_k^T, X \rangle + s_k = D_k^2, \quad k = 1, \dots, K \\
& \mu_k^T x + \hat{s}_k = D_k, \quad \sum_{k=1}^{K} z_k = 1, \quad k = 1, \dots, K \\
& U^+ X_{ij} - Z^k_{i,j} - Z^{k'}_{i,j} = 0, \quad \hat{F}_k - Z^k_{i,j} - Z^{k''}_{i,j} = 0, \quad i,j = 1, \dots, n, k = 1, \dots, K \\
& Z^k_{i,j} - \hat{F}_k - X_{i,j} U^+ - Z^{k'''}_{i,j} = -U^+, \quad i,j = 1, \dots, n, k = 1, \dots, K \\
& \hat{F}_k - b_l z_k - \hat{s}_{kl} = a_l, \quad l = 1, \dots, N, k = 1, \dots, K \\
(AP) \quad & x_i - X_{ij} - X'_{ij} = 0, \quad x_j - X_{ij} - X''_{ij} = 0, \quad i,j = 1, \dots, n \qquad (6) \\
& x_i + x_j - 1 - X_{ij} + X'''_{ij} = 0, \quad i,j = 1, \dots, n \\
& x_i - X_{ii} = 0, \quad i = 1, \dots, n \\
& w_t^T x = d_t, \quad t = 1, \dots, m \\
& \bar{w}_{\bar{t}}^T x + \bar{s}_{\bar{t}} = \bar{d}_{\bar{t}}, \quad \bar{t} = 1, \dots, \bar{m} \\
& z_k, s_k, \hat{s}_k, \bar{s}_{\bar{t}} \geq 0, \quad s_{kl} \geq 0, \quad Z^k_{i,j}, Z^{k'}_{i,j}, Z^{k''}_{i,j}, Z^{k'''}_{i,j} \geq 0 \\
& X_{ij}, X'_{ij}, X''_{ij}, X'''_{ij} \geq 0, \quad i,j = 1, \dots, n \\
& x \in \{0,1\}^n
\end{aligned}
$$

- For the sake of simplicity, problem (6) can be rewritten as:

$$\begin{array}{ll} \max & \hat{c}^T y \\ s.t. & \hat{w}_t y = \hat{d}_t, \quad t = 1, \ldots, K(3n^2 + N + 2) + 3n^2 + n + m + \bar{m} + 1 \\ & y_i \in \{0, 1\}, \quad i = 1, \ldots, n \\ & y \in \mathbb{R}_+^{(4n^2+N+3)K+4n^2+n+\bar{m}} \end{array} \tag{7}$$

where $\hat{c}$, $\hat{w}_t$ and $\hat{d}_t$ are defined accordingly.

Theorem

(AP) is equivalent to a completely positive problem as follows:

$$\begin{array}{lll} OPT_{CP} = \max & \hat{c}^T y & \\ s.t. & \hat{w}_t y = \hat{d}_t, \quad t = 1, \ldots, K(3n^2 + N + 2) + 3n^2 + n + m + \bar{m} + 1 \\ & \langle \hat{w}_t \hat{w}_t^T, Y \rangle = \hat{d}_t^2, \quad t = 1, \ldots, K(3n^2 + N + 2) + 3n^2 + n + m + \bar{m} + 1 \\ (CP) & y_i = Y_{ii}, \quad i = 1, \ldots, n \\ & \begin{pmatrix} 1 & y^T \\ y & Y \end{pmatrix} \in \mathcal{C}^*_{(4n^2+N+3)K+4n^2+n+\bar{m}+1} \end{array}$$

Proof.

Firstly, we prove that the constraints of (AP) implies that $x_i \leq 1$.
Because of the constraints $X_{ij} \leq x_j$ and $X_{ij} \geq x_i + x_j - 1$, $i, j = 1, \ldots, n$, it is obvious that $x_i - 1 \leq 0$. In a second step, one applies the theory presented in Burer (2009). □

Computational study

- In this section we present some computational results to illustrate the efficiency of our new CP formulation introduced in the previous section.
- As optimizing over the cone of completely positive matrices is difficult cf. Murty (1987), a typical approach is to optimize over simpler and more tractable cones.
- We use the inner approximations proposed by Parrilo (2000), namely the first cone $(\mathbb{S}_+ + \mathbb{N})$ where $\mathbb{S}_+$ is the cone of semidefinite matrices and $\mathbb{N}$ is the cone of nonnegative matrices.
- Before discussing our numerical results, we give an SDP approximation by using the piecewise tangent method cf. Cheng et L. (2012) whose value is an upper bound of LPJPC.

- By using the piecewise tangent approximation of $F^{-1}(p^{z_k})$, we have the approximation of (P_1) as follows:

Approximation

$$
\begin{aligned}
\max \quad & c^T x \\
s.t. \quad & \mu_k^T x + ||\Sigma_k^{1/2} \tilde{z}_k|| \leq D_k, \quad k = 1, \ldots, K \\
& \tilde{z}_{ki} \geq a_l x_i + b_l z_{ki}, \quad l = 0, 1, \ldots, N, i = 1, \ldots, n \\
& \sum_{k=1}^{K} z_{ki} = x_i, \quad z_{ki} \geq 0, \quad i = 1, \ldots, n, k = 1, \ldots, K \\
& \tilde{z}_{ki} \leq \tilde{F}_k, \quad \tilde{z}_{ki} \leq M^+ x_i, \quad \tilde{z}_{ki} \geq 0 \\
& \tilde{z}_{ki} \geq \tilde{F}_k - (1 - x_i) M^+, \quad i = 1, \ldots, n, k = 1, \ldots, K \\
& \tilde{F}_k \geq a_l + b_l z_k \quad l = 0, 1, \ldots, N, \\
& \sum_{k=1}^{K} z_k = 1, \quad z_k \geq 0, k = 1, \ldots, K \\
& w_t^T x = d_t, \quad t = 1, \ldots, m \\
& \bar{w}_{\bar{t}}^T x \leq \bar{d}_{\bar{t}}, \quad \bar{t} = 1, \ldots, \bar{m} \\
& x \in \{0, 1\}^n
\end{aligned}
\tag{8}
$$

where $a_0 = 0$, $b_0 = 0$ and $\tilde{z}_k = (\tilde{z}_{k1}, \ldots, \tilde{z}_{kn})$, M^+ is an upper bound of $\tilde{F}_k$ and $\tilde{F}_k = \max_{l=0,\ldots,N}\{a_l + b_l \cdot z_k\}$ is a piecewise tangent approximation of $F^{-1}(p^{z_k})$.

Theorem

The optimal value of (8) is an upper bound of LPJPC. Furthermore, it converges to the optimal value of LPJPC, as the number of segments N goes to infinity.

Proof.

The proof here relies on applying the standard linearization technique cf. Fortet (1959), the theory presented in Cheng et L. (2012) and the results of the piecewise linear approximation presented in Thakur (1978). □

- By applying piecewise tangent approximations of $F^{-1}(p^{z_k})$ and $\frac{1}{F^{-1}(p^{z_k})}$, we get the approximation of (P_0) as follows:

$$
\begin{aligned}
\max \quad & c^T x \\
s.t. \quad & ||\Sigma_k^{1/2} x|| \leq \hat{a}_l D_k - \hat{a}_l u_k^T x + \hat{b}_l D_k z_k - \hat{b}_l u_k^T \hat{z}_k, \quad l = 0, 1, \ldots, N \\
& ||\Sigma_k^{1/2} \tilde{z}_k|| \leq D_k - \mu_k^T x, \quad k = 1, \ldots, K \\
& \tilde{z}_{ki} \geq a_l x_i + b_l z_{ki}, \quad l = 0, 1, \ldots, N, i = 1, \ldots, n \\
& \sum_{k=1}^{K} z_{ki} = x_i, \quad z_{ki} \geq 0, \quad i = 1, \ldots, n, k = 1, \ldots, K \\
& \tilde{z}_{ki} \leq \tilde{F}_k, \quad \tilde{z}_{ki} \leq M^+ x_i, \quad \tilde{z}_{ki} \geq 0 \\
& \tilde{z}_{ki} \geq \tilde{F}_k - (1 - x_i) M^+, \quad i = 1, \ldots, n, k = 1, \ldots, K \\
& \tilde{F}_k \geq a_l + b_l z_k \quad l = 0, 1, \ldots, N, \\
& \sum_{k=1}^{K} z_k = 1, \quad z_k \geq 0, k = 1, \ldots, K \\
& w_t^T x = d_t, \quad t = 1, \ldots, m \\
& \bar{w}_{\bar{t}}^T x \leq \bar{d}_{\bar{t}}, \quad \bar{t} = 1, \ldots, \bar{m} \\
& x \in \{0, 1\}^n
\end{aligned}
\tag{9}
$$

where $a_0 = 0$, $b_0 = 0$ and $\hat{z}_k = (z_{k1}, \ldots, z_{kn})$, M^+ is an upper bound of $\tilde{F}_k$, and $\tilde{F}_k = \max_{l=0,\ldots,N}\{a_l + b_l \cdot z_k\}$ is a piecewise tangent approximation of $F^{-1}(p^{z_k})$.

Theorem

If $p \geq F^{-1}(\sqrt{2})$, then the optimal value of problem (9) is an upper bound of LPJPC. Furthermore, it converges to the optimal value of LPJPC, as the number of segments N goes to infinity.

Proof.

First, we prove that $G(z) := \frac{1}{F^{-1}(p^z)}$ is concave on the interval (0,1] when $p \geq F^{-1}(\sqrt{2})$.
The second derivative of $G(z)$ is given by

$$G^{''}(z) = \frac{-(\ln p)^2 p^z F^{-1}(p^z)[F^{-1}(p^z)f(F^{-1}(p^z)) - 2p^z + p^z(F^{-1}(p^z))^2]}{(F^{-1}(p^z))^4(f(F^{-1}(p^z)))^2}$$

When $p > 0.5$, $F^{-1}(p^z)$ is positive. Then, in order to check the positiveness, it is sufficient to show this property for the term

$$(F^{-1}(p^z))^2 - 2.$$

Therefore, as $F^{-1}(p^z) \geq \sqrt{2}$, i.e., $p^z \geq F(\sqrt{2})$, $G^{''}(z)$ is non-positive. Thus, if $p \geq F(\sqrt{2})$, then $\frac{1}{F^{-1}(p^z)}$ is concave on the interval $(0, 1]$. Then we apply the theory presented in Cheng et L. (2012). □

A Semidefinite relaxation of (9) can be written as:

$$
\begin{aligned}
\max \quad & c^T x \\
s.t. \quad & \begin{pmatrix} v_k I & \Sigma_k^{1/2} x \\ x^T (\Sigma_k^{1/2})^T & v_k \end{pmatrix} \succeq 0, \quad l = 0, 1, \ldots, N, k = 1, \ldots, K \\
& v_k \le D_k - \hat{a}_l u_k^T x + \hat{b}_l D_k z_k - \hat{b}_l u_k^T \hat{z}_k, \quad l = 0, 1, \ldots, N, k = 1, \ldots, K \\
& \begin{pmatrix} (D_k - \mu_k^T x) I & \Sigma_k^{1/2} \tilde{z}_k \\ \tilde{z}_k^T (\Sigma_k^{1/2})^T & D_k - \mu_k^T x \end{pmatrix} \succeq 0, \quad k = 1, \ldots, K \\
& \tilde{z}_{ki} \ge a_l x_i + b_l z_{ki}, \quad l = 0, 1, \ldots, N, i = 1, \ldots, n \\
& \sum_{k=1}^{K} z_{ki} = x_i, \quad z_{ki} \ge 0, \quad i = 1, \ldots, n, k = 1, \ldots, K \\
& \tilde{z}_{ki} \le \tilde{F}_k, \quad \tilde{z}_{ki} \le M^+ x_i, \quad \tilde{z}_{ki} \ge 0 \\
& \tilde{z}_{ki} \ge \tilde{F}_k - (1 - x_i) M^+, \quad i = 1, \ldots, n, k = 1, \ldots, K \\
& \tilde{F}_k \ge a_l + b_l z_k \quad l = 0, 1, \ldots, N, \\
& \sum_{k=1}^{K} z_k = 1, \quad z_k \ge 0, k = 1, \ldots, K \\
& w_t^T x = d_t, \quad w_t^T X w_t = d_t^2, \quad t = 1, \ldots, m \\
& \bar{w}_{\bar{t}}^T x \le \bar{d}_{\bar{t}}, \quad \bar{t} = 1, \ldots, \bar{m} \\
& \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0, X \ge 0.
\end{aligned}
\tag{10}
$$

Numerical results

- We test our formulation SDP relaxations on stochastic multidimentional knapsack problems (SMKP for short) from the OR-library Beasley (2010).
- Instance sizes are represented by three parameters: number of variables (n), number of deterministic constraints (m), and the number of joint chance constraints (K). We consider four problem sizes, i.e., $(n, m, K) = (28, 10, 5); (28, 10, 10); (50, 5, 5); (50, 5, 10)$.
- The probabilistic capacity constraints are generated with vector means μ_k drawn from the uniform distribution on $[0, 5]$, and the covariance matrix Σ_k generated by MATLAB function "gallery('randcor',n)*2".
- The capacity D is independently chosen from $[10, 20]$ interval. Confidence parameter is set to $\alpha = 0.05$.

- We solve and compare three relaxations.

  1. The first one is SDP relaxation whose solution objective value is designed hereafter by V^{SDP}.
  2. The second one solves (AP_0) which is a mixed integer problem and designed by V^{MILP}.
  3. We also implement the continuous linear relaxation of (AP_0), called hereafter V^{LP}. For all three approximations, we choose three tangent points $z_1 = 0.01$, $z_2 = 0.15$ and $z_3 = 0.45$.

- All the considered models are generated using MATLAB environment and solved either by IBM CPLEX $v12$ on an Intel(R)D @ 2.00 GHz with 4.0 GB RAM, or by Sedumi with default parameters.

	K=5							
	V^{MILP}	CPU (s)	V^{LP}	CPU (s)	Gap(%)	V^{SDP}	CPU(s)	GAP(%)
Inst01	6.75e+03	44.81	8.83e+03	2.76	30.81	7.15e+03	5.53	5.93
Inst02	6.95e+03	125.84	9.84e+03	2.23	41.58	7.52e+03	4.78	8.20
Inst03	5.65e+03	37.71	7.52e+03	2.32	33.10	6.09e+03	6.12	7.79
Inst04	6.60e+03	64.09	8.95e+03	2.25	35.61	7.15e+03	4.24	8.33
Inst05	7.17e+03	32.58	8.74e+03	1.93	21.90	7.72e+03	4.80	7.67
Average	–	61.00	–	2.30	32.60	–	5.09	7.58
	K=10							
	V^{MILP}	CPU (s)	V^{LP}	CPU (s)	Gap(%)	V^{SDP}	CPU(s)	GAP(%)
Inst11	5.73e+03	108.97	7.40e+03	8.67	29.14	6.35e+03	14.60	10.82
Inst12	5.65e+03	165.09	7.20e+03	6.68	27.43	5.66e+03	18.45	0.18
Inst13	6.92e+03	177.26	8.39e+03	7.75	21.24	7.04e+03	15.96	1.73
Inst14	6.05e+03	69.88	8.05e+03	5.14	33.06	6.66e+03	13.82	10.08
Inst15	5.65e+03	145.90	7.70e+03	9.19	36.28	6.26e+03	14.39	10.80
Average	–	133.42	–	7.49	29.43	–	15.44	6.72

Table: Computational results for n=28

K=5								
	V^{MILP}	CPU (s)	V^{LP}	CPU (s)	Gap(%)	V^{SDP}	CPU(s)	GAP(%)
Inst21	9.42e+03	739.22	1.22e+04	7.56	29.51	9.65e+03	15.18	2.55
Inst22	9.13e+03	1026.08	1.20e+04	7.01	31.43	9.50e+03	15.39	3.89
Inst23	1.01e+04	371.84	1.22e+04	6.25	20.79	1.03e+04	15.32	1.98
Inst24	8.17e+03	1075.77	1.26e+04	6.12	54.22	9.24e+03	15.80	13.10
Inst25	9.13e+03	1122.23	1.20e+04	7.15	31.43	9.74e+03	15.38	6.68
Average	–	867.03	–	6.82	33.48	–	15.41	4.08
K=10								
	V^{MILP}	CPU (s)	V^{LP}	CPU (s)	Gap(%)	V^{SDP}	CPU(s)	GAP(%)
Inst31	7.88e+03	3080.24	1.08e+04	84.90	37.06	9.04e+03	259.42	14.72
Inst32	7.32e+03	3967.43	1.15e+04	65.81	57.10	8.62e+03	318.80	17.76
Inst33	7.32e+03	863.45	9.57e+03	56.51	30.74	8.02e+03	243.24	9.56
Inst34	7.32e+03	4610.71	1.12e+04	90.92	53.01	8.59e+03	278.38	17.35
Inst35	7.75e+03	2623.89	1.10e+04	55.73	41.94	9.01e+03	270.24	16.26
Average	–	3033.10	–	70.77	43.97	–	270.04	15.13

Table: Computational results for n=50

DATA			MILP		LP+B& B			SDP+B& B		
Name	n	K	"Opt-val"	CPU (s)	UB1	CPU (s)	Nodes	UB2	CPU (s)	Nodes
Inst1	28	5	6.75e+03	44.81	6.75e+03	622.55	2179	6.75e+03	397.13	111
Inst2	28	5	6.95e+03	125.84	6.95e+03	1504.51	5788	6.95e+03	405.10	109
Inst3	28	5	5.65e+03	37.71	5.65e+03	119.19	425	5.65e+03	233.62	56
Inst4	28	5	6.60e+03	64.09	6.60e+03	427.81	1372	6.60e+03	233.68	64
Inst5	28	5	7.17e+03	32.58	7.17e+03	319.45	1357	7.17e+03	228.97	65
Inst6	28	10	5.73e+03	108.97	5.73e+03	943.58	846	5.73e+03	821.11	68
Inst7	28	10	5.65e+03	165.09	5.65e+03	876.85	674	5.65e+03	69.05	4
Inst8	28	10	6.92e+03	177.26	6.92e+03	1369.65	1267	6.92e+03	763.69	57
Inst9	28	10	6.05e+03	69.88	6.05e+03	746.91	838	6.05e+03	810.34	66
Inst10	28	10	5.65e+03	145.90	5.65e+03	2364.66	2712	5.65e+03	657.19	56
Inst11	50	5	9.42e+03	739.22	9.42e+03	11744.74	9943	9.42e+03	1428.76	102
Inst12	50	5	9.13e+03	1026.08	9.13e+03	38829.47	31192	9.13e+03	1741.27	124
Inst13	50	5	1.01e+04	371.84	1.01e+04	7570.27	5135	1.01e+04	1283.81	84
Inst14	50	5	8.17e+03	1075.77	8.17e+03	16968.03	15567	8.17e+03	1512.26	97
Inst15	50	5	9.13e+03	1122.33	9.13e+03	8909.88	6638	9.13e+03	1577.06	102
Inst16	50	10	7.88e+03	3080.24	7.88e+03	37629.30	9430	7.88e+03	23011.53	104

Conclusions

- New CP formulation for the joint probabilistic $0-1$ LP problems.
- Derive tight SDP relaxations for the CP problem.
- Preliminary numerical results show the CP formulation is promising.
- Applications to Energy management bring new insight for solving important practical problem.