Max-plus numerical methods for Hamilton-Jacobi equations: attenuation of the curse of dimensionality

Zheng QU

CMAP and INRIA, École Polytechnique

PGMO project: Tropical methods in optimization

October 4, 2013

Talk based on the work of Fleming, McEneaney, Akian, Gaubert, etc and the PhD work of Z.QU.

Deterministic optimal control problem

- State space $X \subset \mathbb{R}^d$, control space U
- Value function at time T:

$$V_T(x) := \sup_{\mathbf{u} \in \mathcal{U}_T} \int_0^T \ell(\mathbf{x}(s), \mathbf{u}(s)) ds + \phi(\mathbf{x}(T))$$

with state dynamics:

$$\dot{\mathbf{x}}(s) = f(\mathbf{x}(s), \mathbf{u}(s)), \quad s \in [0, T)$$
$$\mathbf{x}(0) = x.$$

where

$$\mathcal{U}_T := \{\mathbf{u} \in L^2([0, T]; U)\}.$$

Optimal control problem/Dynamic programming

- Dynamic programming principle:

$$V_{t+\tau} = S_\tau[V_t], \quad \forall t \geq 0, 0 \leq \tau \leq T - t$$

where the Lax-Oleinik semigroup $(S_t)_{t\geq 0} : \mathcal{F} \to \mathcal{F}$ is defined as:

$$S_t[\phi](x) := \sup_{\mathbf{u}} \int_0^t \ell(\mathbf{x}(s), \mathbf{u}(s))ds + \phi(\mathbf{x}(t))$$

with the state dynamic:

$$\dot{\mathbf{x}}(s) = f(\mathbf{x}(s), \mathbf{u}(s)), \quad s \in [0, T)$$
$$\mathbf{x}(0) = x.$$

Optimal control problem/HJ PDE

- Dynamic Programming Principle $\rightarrow$ HJ PDE:

$$\frac{\partial V_t}{\partial t} - H(x, \frac{\partial V_t}{\partial x}) = 0, \qquad V_0 = \phi$$

where H is the Hamiltonian of the system given by:

$$H(x,p) := \sup_{u \in U} < p, f(x,u) > + \ell(x,u).$$

- Under certain regularity assumptions, the value function $V_{\cdot}(\cdot)$ is a (unique) viscosity solution of the above HJ PDE.

Classical numerical methods

- Finite difference methods
  - Discretize $t \in [0, T]$ and $x \in X \subset \mathbb{R}^d$
  - Approximate $\frac{\partial V_t}{\partial t}$ and $\frac{\partial V_t}{\partial x}$ using a finite difference scheme on discretized points.
- Semi-Lagrangian methods ([Falcone])
  - Discretize $t \in [0, T]$ and $x \in X \subset \mathbb{R}^d$
  - Approximate the dynamic programming equation by

$$V_{t+\Delta t}(x) = \sup_{u \in U} \{\Delta t \ell(x, u) + V_t(x + \Delta t f(x, u))\}$$

  - Obtain the value function by an interpolation scheme on discretized points.

From classical methods to max-plus basis methods

- In classical methods, the value function is approximated by a linear combination of simpler functions:

$$V(x) \simeq \sum_{i=1}^{n} \lambda_i \mathbf{w}_i(x)$$

The coefficients $\{\lambda_i\}_i$ are obtained by interpolation on dicretized points.

- The max-plus basis method, first developped in [Fleming,McEneaney 00], approximates the value function by a max-plus linear combination of basis functions:

$$V(x) \simeq \bigoplus_i \lambda_i \otimes \mathbf{w}_i(x) = \sup_i \lambda_i + \mathbf{w}_i(x)$$

The coefficients $\{\lambda_i\}_i$ are obtained by propagation of the Lax-Oleinik semigroup, no requirement on the state space discretization.

Max-plus/tropical algebra

- Max-plus semiring

$$\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$$
$$a \oplus b := \max(a, b), \quad a \otimes b := a + b$$

- Classical linear combination:

$$V(x) \simeq \sum_{i=1}^{n} \lambda_i \mathbf{w}_i(x)$$

- Max-plus linear combination:

$$V(x) \simeq \bigoplus_i \lambda_i \otimes \mathbf{w}_i(x)$$
$$= \sup_i \mathbf{w}_i(x) + \lambda_i$$

Max-plus linearity of Lax-Oleinik semigroup

- $\forall t \geq 0$, $\lambda \in \mathbb{R}_{\max}$, $\phi, \psi \in \mathcal{F}$,

$$S^t[\sup(\phi,\psi)] = \sup(S^t[\phi], S^t[\psi])$$
$$S^t[\lambda + \phi] = \lambda + S^t[\phi]$$

Max-plus linearity of Lax-Oleinik semigroup

- $\forall t \geq 0,\ \lambda \in \mathbb{R}_{\max},\ \phi, \psi \in \mathcal{F}$,

$$S^t[\sup(\phi,\psi)] = \sup(S^t[\phi], S^t[\psi])$$
$$S^t[\lambda + \phi] = \lambda + S^t[\phi]$$

- Max-plus linearity: $\forall t \geq 0,\ \lambda \in \mathbb{R}_{\max},\ \phi, \psi \in \mathcal{F}$,

$$S^t[\phi \oplus \psi] = S^t[\phi] \oplus S^t[\psi]$$
$$S^t[\lambda \otimes \phi] = \lambda \otimes S^t[\phi]$$

Max-plus basis method: general principle

- Chose a set of adapted basis functions

$$\mathcal{B} = \{\mathbf{w}_i\}_{i \in I}$$

- Discretize over time interval $[0, T]$ by time step τ:

$$t = 0, \tau, 2\tau, \ldots, N\tau.$$

- Approximate by propagation the value functions $\{V_\tau, V_{2\tau}, \ldots, V_{N\tau}\}$ on the subsemimodule (subspace) generated by $\mathcal{B}$

$$\mathrm{Span}\,\mathcal{B} := \{\bigoplus_{i \in I} \lambda_i \otimes \mathbf{w}_i : \lambda \in \mathbb{R}^I_{\max}\}$$

Max-plus basis method: general principle

- Chose a set of adapted basis functions

$$\mathcal{B} = \{\mathbf{w}_i\}_{i\in I}$$

- Discretize over time interval $[0, T]$ by time step τ:

$$t = 0, \tau, 2\tau, \ldots, N\tau.$$

- Approximate by propagation the value functions $\{V_\tau, V_{2\tau}, \ldots, V_{N\tau}\}$ on the subsemimodule (subspace) generated by $\mathcal{B}$

$$\begin{aligned}\operatorname{Span}\mathcal{B} &:= \{\bigoplus_{i\in I}\lambda_i \otimes \mathbf{w}_i : \lambda \in \mathbb{R}^I_{\max}\} \\ &= \{\sup_{i\in I}\lambda_i + \mathbf{w}_i : \lambda \in \mathbb{R}^I_{\max}\}.\end{aligned}$$

Propagation principle

- Propagation principle
  - At time t,
$$V_t \simeq \tilde{V}_t = \sup_{i \in I} \lambda_i^t + \mathbf{w}_i \in \operatorname{Span} \mathcal{B}.$$
  - At time $t+\tau$,
$$V_{t+\tau} = S^{\tau}[V_t] \quad \text{by dynamic programming}$$

Propagation principle

- Propagation principle
  - At time t,

$$V_t \simeq \tilde{V}_t = \sup_{i \in I} \lambda_i^t + \mathbf{w}_i \in \operatorname{Span} \mathcal{B}.$$

  - At time $t+\tau$,

$$\begin{aligned} V_{t+\tau} &= S^{\tau}[V_t] \quad \text{by dynamic programming} \\ &\simeq S^{\tau}[\tilde{V}_t] \end{aligned}$$

Propagation principle

- Propagation principle
  - At time t,

$$V_t \simeq \tilde{V}_t = \sup_{i \in I} \lambda_i^t + \mathbf{w}_i \in \operatorname{Span} \mathcal{B}.$$

  - At time $t + \tau$,

$$\begin{aligned} V_{t+\tau} &= S^\tau[V_t] \quad \text{by dynamic programming} \\ &\simeq S^\tau[\tilde{V}_t] \\ &= S^\tau[\sup_{i \in I} \lambda_i^t + \mathbf{w}_i] \end{aligned}$$

Propagation principle

- Propagation principle
  - At time t,

$$V_t \simeq \tilde{V}_t = \sup_{i\in I} \lambda_i^t + \mathbf{w}_i \in \operatorname{Span} \mathcal{B}.$$

  - At time $t+\tau$,

$$\begin{aligned} V_{t+\tau} &= S^\tau[V_t] \quad \text{by dynamic programming} \\ &\simeq S^\tau[\tilde{V}_t] \\ &= S^\tau[\sup_{i\in I} \lambda_i^t + \mathbf{w}_i] \\ &= \sup_{i\in I} \lambda_i^t + S^\tau[\mathbf{w}_i] \quad \text{by max} - \text{plus linearity of } S^\tau \end{aligned}$$

Propagation principle

- Propagation principle
  - At time t,

$$V_t \simeq \tilde{V}_t = \sup_{i\in I} \lambda_i^t + \mathbf{w}_i \in \operatorname{Span}\mathcal{B}.$$

  - At time $t+\tau$,

$$\begin{aligned} V_{t+\tau} &= S^\tau[V_t] \quad \text{by dynamic programming} \\ &\simeq S^\tau[\tilde{V}_t] \\ &= S^\tau[\sup_{i\in I} \lambda_i^t + \mathbf{w}_i] \\ &= \sup_{i\in I} \lambda_i^t + S^\tau[\mathbf{w}_i] \quad \text{by max} - \text{plus linearity of } S^\tau \\ &\simeq \sup_{i\in I} \lambda_i^t + \tilde{S}^\tau[\mathbf{w}_i] \ \ (\text{semigroup approximation}) \end{aligned}$$

Propagation principle

- Propagation principle
  - At time t,

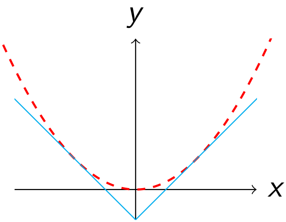
$$V_t \simeq \tilde{V}_t = \sup_{i \in I} \lambda_i^t + \mathbf{w}_i \in \mathrm{Span}\,\mathcal{B}.$$

  - At time $t+\tau$,

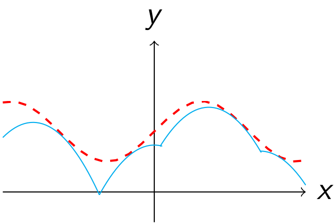
$$\begin{aligned} V_{t+\tau} &= S^\tau[V_t] \quad \text{by dynamic programming} \\ &\simeq S^\tau[\tilde{V}_t] \\ &= S^\tau[\sup_{i \in I} \lambda_i^t + \mathbf{w}_i] \\ &= \sup_{i \in I} \lambda_i^t + S^\tau[\mathbf{w}_i] \quad \text{by max} - \text{plus linearity of } S^\tau \\ &\simeq \sup_{i \in I} \lambda_i^t + \tilde{S}^\tau[\mathbf{w}_i] \ \ (\text{semigroup} \ \ \text{approximation}) \\ &\simeq \sup_{i \in I} \lambda_i^{t+\tau} + \mathbf{w}_i \ \ (\text{projection}) \end{aligned}$$

Examples and References

Linear basis functions



Quadratic basis functions



References: [Fleming,McEneaney 00], [Akian,Gaubert,Lakhoua06](finite element max-plus method), [McEneaney 07](a curse of dimensionality free method), [McEneaney,Deshpande,Gaubert 08], [Sridharan,James,McEneaney 10], [Dower,McEneaney 11]...

Infinite horizon switched problem [McEneaney 07]

$$V(x) = \sup_{\substack{\mathbf{u}\in W\\ \mu\in\mathcal{D}_\infty}} \int_0^\infty \frac{1}{2}\mathbf{x}(t)' D^{\mu(t)}\mathbf{x}(t) + (\ell_1^{\mu(t)})'\mathbf{x}(t) + \alpha^{\mu(t)} - \frac{\gamma^2}{2}|\mathbf{u}(t)|^2\, dt$$

where

$$\mathcal{D}_\infty \doteq \{\mu : [0,\infty) \to \{1,\dots,M\} : \text{measurable}\} \ ,$$
$$W \doteq L_2^{\text{loc}}([0,\infty);\mathbb{R}^k) \ ,$$

and the state dynamics are given by

$$\dot{\mathbf{x}}(s) = A^{\mu(s)}\mathbf{x}(s) + \sigma^{\mu(s)}\mathbf{u}(s) + \ell_2^{\mu(s)};\ \ \mathbf{x}(0) = x \in \mathbb{R}^d \ . \quad (1)$$

Arising from nonlinear robust H_∞ control, nonconvex problem. V is the available storage function.

Hamiltonian

- The Hamiltonian of the switched system:

$$H(x,p) = \sup_{m\in 1,\dots,M} H^m(x,p)$$

where

$$H^m(x,p) = \frac{1}{2}x'D^m x + \frac{1}{2}p'\Sigma^m p + (A^m x)'p + (\ell_1^m)'x + (\ell_2^m)'p + \alpha^m .$$

- Approximation of general Hamiltonian $\tilde{H}$ by supremum of quadratic functions:

$$\tilde{H} \simeq \sup_{m\in\{1,\dots,M\}} H^m .$$

Lax-Oleinik semigroups [McEneaney 07]

- The semigroup associated to the switched problem:

$$S^t[\phi](x) = \sup_{\mathbf{u}} \sup_{\mu} \int_0^t \frac{1}{2}\mathbf{x}(t)' D^{\mu(t)} \mathbf{x}(t) + (\ell_1^{\mu(t)})' \mathbf{x}(t) + \alpha^{\mu(t)} - \frac{\gamma^2}{2} |\mathbf{u}(t)|^2 dt + \phi(x(t)).$$

where

$$\dot{\mathbf{x}}(s) = A^{\mu(s)}\mathbf{x}(s) + \sigma^{\mu(s)}\mathbf{u}(s) + \ell_2^{\mu(s)}; \;\; \mathbf{x}(0) = x \in \mathbb{R}^d \;\; .$$

- For all $t \geq 0$,

$$S^t[V] = V$$

- For all $V_0 \leq V$,

$$V = \lim_{T \to +\infty} S^T[V_0] \;\; .$$

Max-plus basis method [McEneaney 07]

- Chose a quadratic function $V_0 \leq V$.
- Finite horizon approximation

$$V(x) \simeq V_T(x) := S^T[V_0](x).$$

- Max-plus basis method to approximate V_T:
  - The basis functions are all the quadratic functions.
  - Discretize the time interval $[0, T]$ into $\{0, \tau, 2\tau, \dots, N\tau\}$.
  - Approximate the value functions $V_\tau, V_{2\tau}, \dots, V_{N\tau}$ by the supremum of quadratic functions inferior to V.

Max-plus propagation [McEneaney 07]

- Propagation principle
  - At time t,

$$V_t \simeq \tilde{V}_t = \sup_{i=1,\dots,q_t} \phi_i^t$$

  - At time $t+\tau$,

$$V_{t+\tau} \simeq S^\tau[\tilde{V}_t] = \sup_{i=1,\dots,q_t} S^\tau[\phi_i^t]$$

Max-plus propagation [McEneaney 07]

- Propagation principle
  - At time t,
$$V_t \simeq \tilde{V}_t = \sup_{i=1,\ldots,q_t} \phi_i^t$$
  - At time $t+\tau$,
$$V_{t+\tau} \simeq S^\tau[\tilde{V}_t] = \sup_{i=1,\ldots,q_t} S^\tau[\phi_i^t]$$
$$\simeq \sup_{i=1,\ldots,q_t} \tilde{S}^\tau[\phi_i^t]$$

Approximation of the semigroup [McEneaney 07]

- Approximate S^τ by:

$$S^\tau \simeq \tilde{S}^\tau = \sup_m S_m^\tau$$

- S_m^t is the semigroup associated to the control problem by letting the switching control μ equal to $m \in \{1, \dots, M\}$:

$$S_m^t[\phi](x) = \sup_{\mathbf{u}} \int_0^t \frac{1}{2}\mathbf{x}(t)' D^m \mathbf{x}(t) + (\ell_1^m)'\mathbf{x}(t) + \alpha^m - \frac{\gamma^2}{2}|\mathbf{u}(t)|^2 dt + \phi(\mathbf{x}(t)).$$

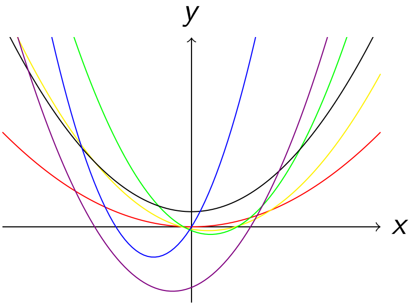
$$\dot{\mathbf{x}}(s) = A^m \mathbf{x}(s) + \sigma^m \mathbf{u}(s) + \ell_2^{\mu(s)}; \ \ \mathbf{x}(0) = x \in \mathbb{R}^d \ .$$

- $S_m^t[\phi]$ is a quadratic affine function if ϕ is. (Riccati)
- No switch allowed inside the intervals $(0, \tau), \dots, ((N-1)\tau, N\tau)$.

Max-plus propagation [McEneaney 07]

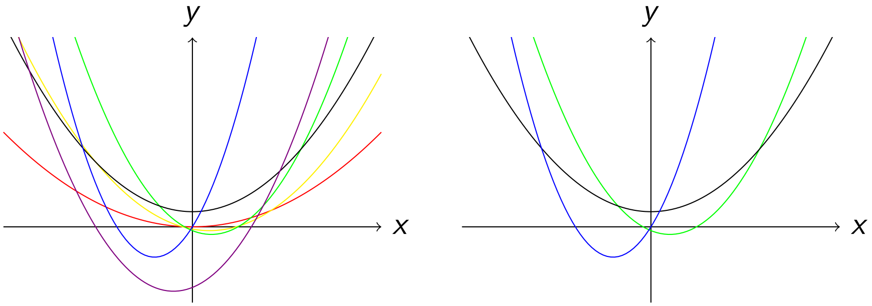
- Propagation principle
  - At time t,
$$V_t \simeq \tilde{V}_t = \sup_{i=1,\dots,q_t} \phi_i^t$$
  - At time $t+\tau$,
$$V_{t+\tau} \simeq S^\tau[\tilde{V}_t] = \sup_{i=1,\dots,q_t} S^\tau[\phi_i^t]$$
$$\simeq \sup_{i=1,\dots,q_t} \sup_{m=1,\dots,M} \underbrace{S_m^\tau[\phi_i^t]}_{Riccati}$$
- At the end of N iterations, the number of quadratic basis functions is M^N.
- The computational complexity $O(M^N d^3)$: curse of dimensionality free, but with a *curse of complexity*
- Such curse of complexity can be reduced by carring on pruning operations.

Pruning operation:



$$\tilde{V} = \sup(\phi_{green}, \phi_{red}, \phi_{violet}, \phi_{yellow}, \phi_{black}, \phi_{blue})$$

Pruning operation:



$$\tilde{V} = \sup(\phi_{green}, \phi_{red}, \phi_{violet}, \phi_{yellow}, \phi_{black}, \phi_{blue})$$

$$\tilde{V} = \sup(\phi_{green}, \phi_{black}, \phi_{blue})$$

SDP based pruning algorithms

Approach in [McEneaney,Deshpande,Gaubert 08]: for every quadratic function ϕ_i, find the *maximum lost* caused by removing it from the basis set, keep those functions with largest maximum lost.

Optimization problem

$$\begin{aligned} \nu_j = \max \ & \nu \\ & \nu \leq z'(Q_j - Q_i)z \\ & z^T z = 1. \end{aligned}$$

SDP relaxation

$$\begin{aligned} \overline{\nu}_j = \max \ & \nu \\ & \nu \leq \mathrm{trace}((Q_j - Q_i)Z) \\ & Z \geq 0, \ \ \mathrm{trace}(Z) = 1. \end{aligned}$$

where $Q_1, \ldots, Q_n$ are $(d+1) \times (d+1)$ matrices such that the quadratic functions are given by

$$\phi_i(x) = (x' \ \ 1) Q_i \begin{pmatrix} x \\ 1 \end{pmatrix}, \quad i = 1, \ldots, n.$$

SDP based pruning algorithms: a refinement

[Gaubert,McEneaney,Qu 11]

- Reduction of pruning to k-median problem
- Generation of witness points:

Optimisation problem

$$\begin{aligned} \nu_j = \max \quad & \nu \\ & \nu \leq z'(Q_j - Q_i)z \\ & z^T z = 1. \end{aligned}$$

SDP relaxation

$$\begin{aligned} \overline{\nu}_j = \max \quad & \nu \\ & \nu \leq \text{trace}((Q_j - Q_i)Z) \\ & Z \geq 0, \ \ \text{trace}(Z) = 1. \end{aligned}$$

Randomization technique [Aspremont,Boyd 03]: For each function j, let Z_j be the optimal solution of the SDP, generate random points of distribution $\mathcal{N}(0, Z_j)$.

- Apply existing algorithms to solve the k-median problem. (Greedy algorithm, Jain-Vazirani algorithm)

Backsubstitution error [McEneaney 07]

For an approximation function $\tilde{V}$, the *backsubstitution error* is:

$$H(x, \nabla \tilde{V}) \ .$$

Under some technical assumptions (on the finiteness of the infinite horizon problem), [McEneaney 07] showed that: the value function V is the unique viscosity solution of

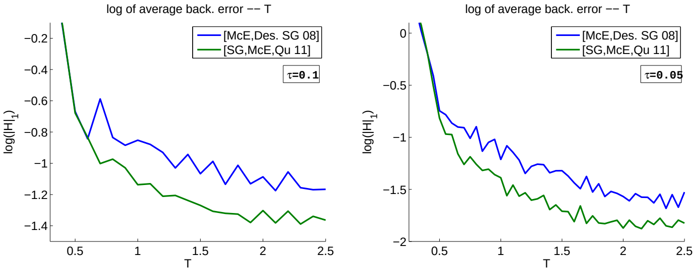
$$-H(x, \nabla V) = 0$$

in a class of bounded functions.

Experimental results

$M = 6$, $d = 6$

Average backsubstitution error evolution

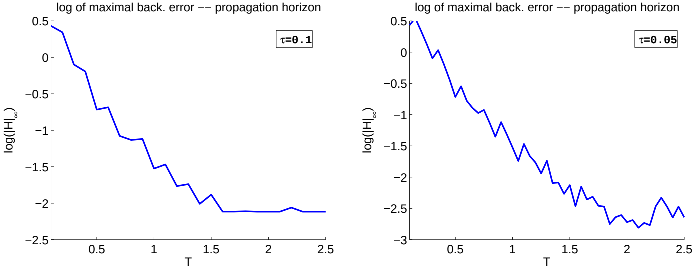


⇒Refined SDP based pruning algorithm works better.

Experimental results

$M = 6, \ d = 6$

Logarithm of the maximal backsubstitution error evolution



$\Rightarrow$ The error decreasing rate seems greater than the error bound $O(1/T)$ established in [McEneaney, Kluberg, SICON'09].

Switched infinite horizon optimal control problem

$$V(x) = \sup_{\mu} \sup_{\mathbf{u}} \int_0^{\infty} \frac{1}{2} \mathbf{x}(t)' D^{\mu(t)} \mathbf{x}(t) - \frac{\gamma^2}{2} |\mathbf{u}(t)|^2 dt,$$

where $\mathbf{x}(\cdot)$ satisfies:

$$\dot{\mathbf{x}}(t) = A^{\mu(t)} \mathbf{x}(t) + \sigma^{\mu(t)} \mathbf{u}(t), \quad \mathbf{x}(0) = x \in \mathbb{R}^d$$

Parameters: $\{A^m, D^m, \sigma^m : \quad m = 1, \dots, M\}$

Assumption (existence)

$$0 \prec D^m \preceq c_D I_d, \quad |\sigma^m| \leq c_\sigma, \forall m = 1, \dots, M$$
$$x' A^m x \leq -c_A |x|^2, \quad \forall x \in \mathbb{R}^d, \forall m = 1, \dots, M$$
$$\gamma^2 / c_\sigma^2 > c_D / c_A^2.$$

Comaprison

[McEneaney, Kluberg, SICON'09]

Under Assumption existence and Assumption σ ($\sigma^m = \sigma$), without pruning error, the total error is:

$$O(\frac{1}{T}) + O(\sqrt{\tau})$$

[Z.Qu 12]

Under Assumption existence and Assumption contraction, if the pruning error is bounded by Δ, then the total error is:

$$O(e^{-\alpha T}) + O(\tau) + O(\frac{\Delta}{\tau})$$

Assumption contraction can be released when $D^m = D$.

Main ingredient: Thompson's metric

- For $A, B \succ 0$, Thomspon's metric is defined by:

$$d_T(A,B) = \| \operatorname{spec}(\log B^{-\frac{1}{2}} A B^{-\frac{1}{2}}) \|_\infty .$$

Thompson's metric is an invariant Finsler metric:

$$d_T(A,B) = \inf_\gamma \int_0^1 \|\dot\gamma(t)\gamma(t)^{-1}\|_\infty dt \qquad \text{Finsler}$$

$$d_T(U^{-1}AU, U^{-1}BU) = d_T(A,B) \qquad \text{invariant}$$

- We extend the definition for two functions $f, g : \mathbb{R}^d \to \mathbb{R}_*^+$ by:

$$d_T(f,g) := \| \log fg^{-1} \|_\infty$$

where $\| \cdot \|_\infty$ is the infinity norm of a function.

Main ingredient

The semigroups act as strict local contractions on a set of bounded quadratic functions:

Theorem ([Z.Qu 12])

Under Assumption existence and Assumption contraction, there is a bounded set of functions $\mathcal{F} \subset G$ invariant under the semigroup actions. Moreover, there is $\alpha > 0$, such that

$$d_T(\tilde{S}^t[\phi_1], \tilde{S}^t[\phi_2]) \leq e^{-\alpha t} d_T(\phi_1, \phi_2), \ \forall t \geq 0,$$
$$d_T(S^t[\phi_1], S^t[\phi_2]) \leq e^{-\alpha t} d_T(\phi_1, \phi_2), \ \forall t \geq 0.$$

for all quadratic functions in the invariant set $\mathcal{F}$.

Recall:

$$\tilde{S}^t[\phi] := \sup_m S^t_m[\phi].$$

A characterization of contraction rate

Consider the following equation defined on $\hat{S}_d^+$:

$$\dot{P} = \Phi(P).$$

The flow is order-preserving if

$$P_1(0) \preceq P_2(0) \Rightarrow P_1(t) \preceq P_2(t), \quad t \geq 0.$$

Theorem ([Gaubert, Qu 12])

If the flow is order-preserving, then the best constant such that

$$d_T(P_1(t), P_2(t)) \leq e^{-\alpha t} d_T(P_1(0), P_2(0)), \quad \forall t \geq 0$$

holds for all solutions contained in an open set $U \subset S_d^+$ such that $\lambda U \subset U$ for $\lambda \in (0,1]$ is given by:

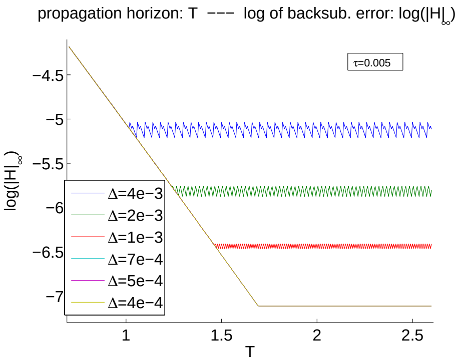
$$\alpha = -\sup_{P \in U} \lambda_{\max}(P^{-1}(D\Phi(P) \circ P - \Phi(P)))$$

A numerical illustration

$d = 2, M = 3.$

time step: τ; pruning error: Δ; propagation horizon: T.

$$O(e^{-\alpha T}) + O(\tau) + O(\Delta/\tau)$$

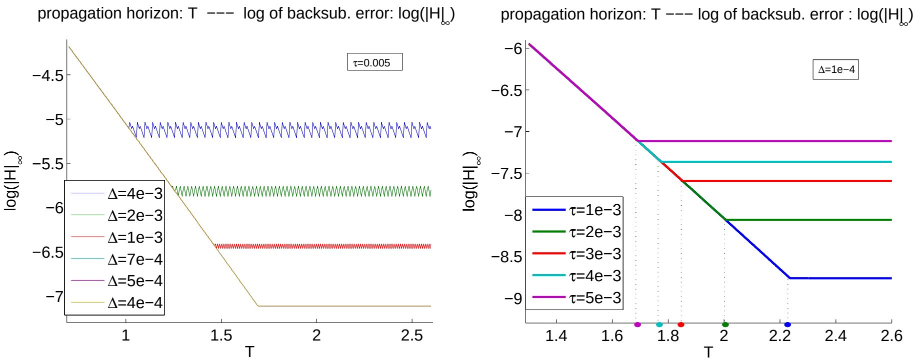


A numerical illustration

$d = 2, M = 3.$

time step: τ; pruning error: Δ; propagation horizon: T.

$$O(e^{-\alpha T}) + O(\tau) + O(\Delta/\tau)$$



Motivation

- SDP based pruning:

$$V \simeq \sup_{i_1,\cdots i_N} S^{\tau}_{i_N} \cdots S^{\tau}_{i_1}[V_0] \ .$$

add all the possible basis functions + prune

- SDP based pruning algorithm works but is time-consuming:

τ=0.2, K=25	Total time	Propagation	SDP	Pruning
sort lower	1.04h	1.85%	98.15%	0.00%
sort upper	1.34h	1.52%	98.43%	0.05%
J-V p-d	1.38h	1.45%	89.47%	9.08%
greedy	1.43h	1.63%	97.84%	0.53%

- New algorithm:

add "*useful*" basis functions (no need to prune)

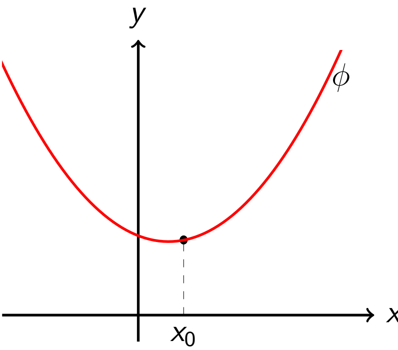
Key observation

Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a quadratic function, $x_0 \in \mathbb{R}^d$ and $m_0 \in \{1, \ldots, M\}$. If

$$H^{m_0}(x_0, D\phi(x_0)) > 0 \quad ,$$

then for all sufficiently small $t > 0$,

$$S^t_{m_0}[\phi](x_0) > \phi(x_0) \quad .$$



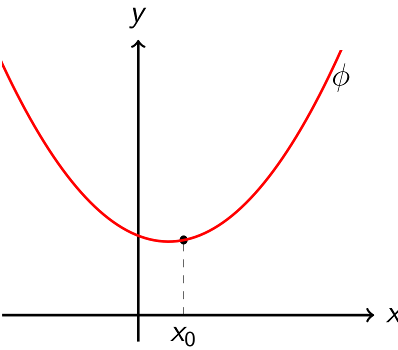
Key observation

Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a quadratic function, $x_0 \in \mathbb{R}^d$ and $m_0 \in \{1, \dots, M\}$. If

$$H^{m_0}(x_0, D\phi(x_0)) > 0 \ ,$$

then for all sufficiently small $t > 0$,

$$S^t_{m_0}[\phi](x_0) > \phi(x_0) \ .$$



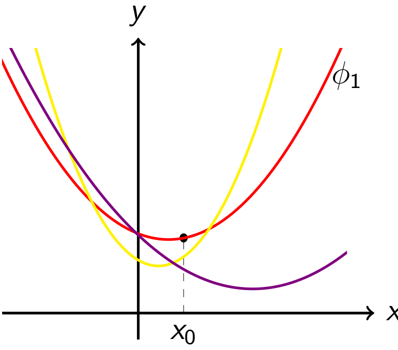
Key observation

Let $\phi_1, \dots, \phi_k : \mathbb{R}^d \to \mathbb{R}$ be quadratic functions and $x_0 \in \mathbb{R}^d$. If

$$\phi_1(x_0) = \sup_{i=1}^k \phi_i(x_0) \quad ,$$
$$\sup_m H^m(x_0, D\phi_1(x_0)) > 0 \quad ,$$

then there is m_0 such that for all sufficiently small $t > 0$,

$$S^t_{m_0}[\phi_1](x_0) > \sup_{i=1}^k \phi_i(x_0) \quad .$$



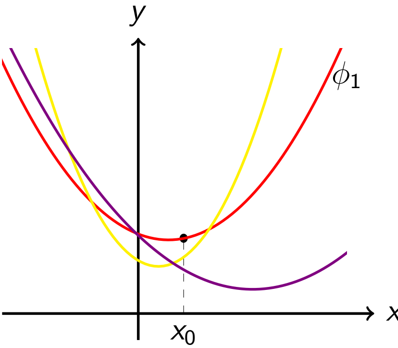
Key observation

Let $\phi_1, \dots, \phi_k : \mathbb{R}^d \to \mathbb{R}$ be quadratic functions and $x_0 \in \mathbb{R}^d$. If

$$\phi_1(x_0) = \sup_{i=1}^k \phi_i(x_0) \quad ,$$
$$\sup_m H^m(x_0, D\phi_1(x_0)) > 0 \quad ,$$

then there is m_0 such that for all sufficiently small $t > 0$,

$$S^t_{m_0}[\phi_1](x_0) > \sup_{i=1}^k \phi_i(x_0) \quad .$$



Randomized max-plus algorithm

Algorithm 1 Randomized max-plus algorithm

INPUT: $V^0 = \sup_i \phi_i^0 \leq V$, compact $X \ni 0$, threshold $\vartheta > 0$
for $k = 0, 1, 2, \ldots$ **do**
  Randomly choose a point $x_0 \in X$;
  if $H(x_0, DV^k(x_0)) > \vartheta$, **then**

$$V^{k+1} = \sup(V^k, \Psi(V^k, x_0));$$

  else

$$V^{k+1} = V^k.$$

  end if
end for

Convergence of the algorithm

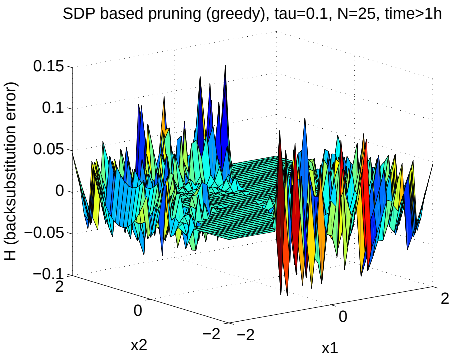
Theorem ([Gaubert, Qu 13])

Under Assumption existence, for everythreshold $\vartheta > 0$, the randomized algorithm stops surely after a finite number of iterations and there is a constant $L > 0$ such that almost surely we have:

$$\lim_{k \to +\infty} \sup_x |V(x) - V^k(x)|/|x|^2 \leq L\vartheta$$

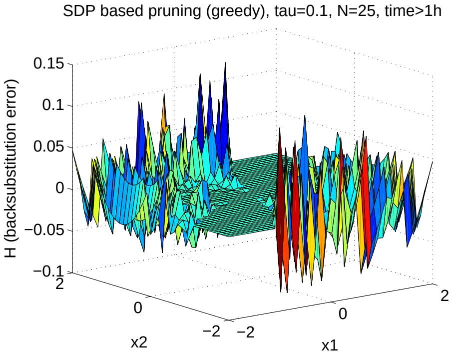
Experimental results

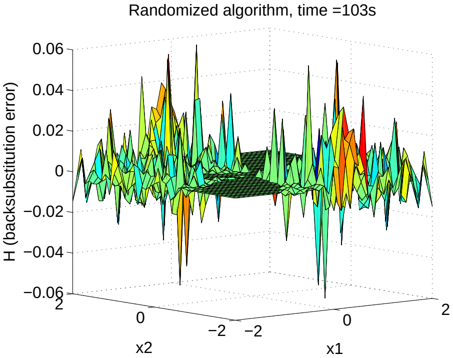
M=6, d=6



Experimental results

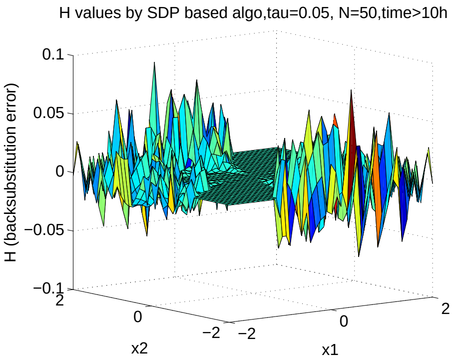
M=6, d=6





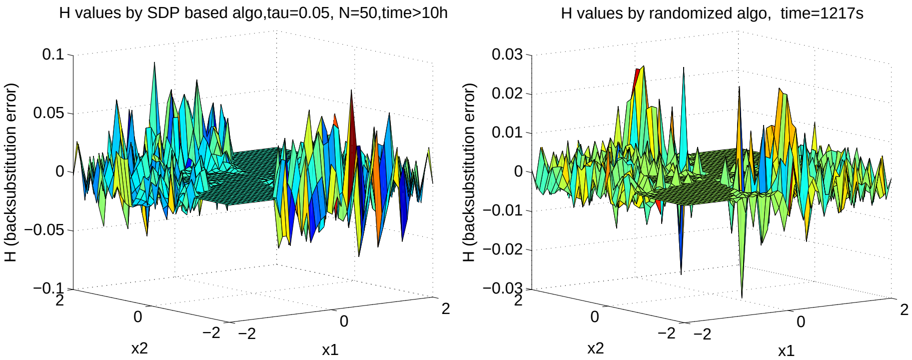
Experimental results

M=6, d=6



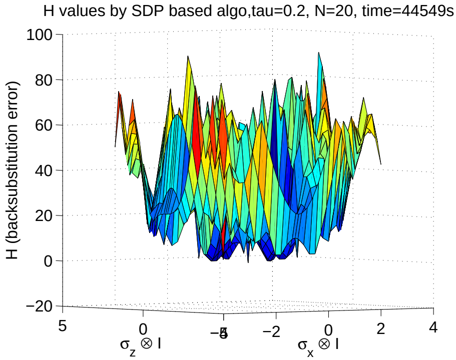
Experimental results

M=6, d=6



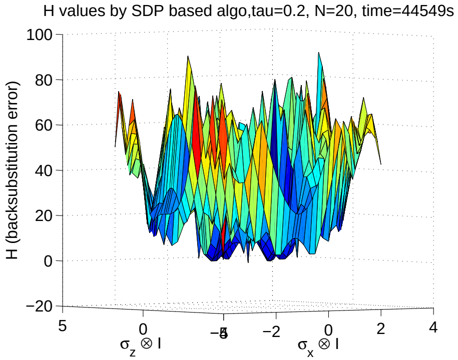
Experimental results

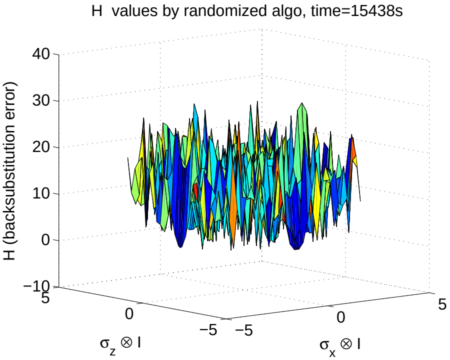
M=6, d=15 (an optimal control problem on SU(4) −quantum optimal gate synthesis)



Experimental results

M=6, d=15 (an optimal control problem on SU(4) −quantum optimal gate synthesis)





Conclusions and perspectives

- Existing work
  - Max-plus basis method: no requirement on the space discretization
  - McEneaney's curse of dimensionality free method
    - curse of dimensionality converted to curse of complexity
    - SDP based pruning algorithm
- Our work
  - A refined SDP based pruning algorithm
  - An improved error bound by applying new results in nonlinear Perron-Frobenius theory
  - A randomized max-plus algorithm showing clear speedup compared to the SDP based algorithms.
- Futur work
  - Apply the COD free approach to more general Hamiltonian $H \simeq \sup_m H^m$

Akian,Gaubert,Lakhoua 06.
The max-plus finite element method for solving deterministic optimal control problems: basic properties and convergence analysis.
SICON. 47(2): 817-848, 2006

W. H. Fleming and W. M. McEneaney.
A max-plus-based algorithm for a Hamilton-Jacobi-Bellman equation of nonlinear filtering
SICON, 38(3):683-710, 2000.

Falcone,M.
A numerical approach to the infinite horizon problem of deterministic control theory
Appl. Math. Optim. 1987: 1, 1-13

W. M. McEneaney, A. Deshpande and S. Gaubert

Curse-of-Complexity Attenuation in the Curse-of-Dimensionality-Free Method for HJB PDEs
Proc. of the 2008 ACC :4684-4690, 2008.

S. Gaubert, W. M. McEneaney and Z. QU
Curse of dimensionality reduction in max-plus based approximation methods: Theoretical estimates and improved pruning algorithms
Proc. of the 2011 CDC :1054-1061, 2011.

W. M. McEneaney
A curse-of-dimensionality-free numerical method for solution of certain HJB PDEs
SICON 46(4):1239-1076, 2007.

W. M. McEneaney
Idempotent algorithms for discrete-time stochastic control through distributed dynamic programming

Proc. of the 2009 CDC :1569-1574, 2009.

H. Kaise , W. M. McEneaney
Idempotent expansions for continuous-time stochastic control: compact control space
Proc. of the 2010 CDC :7015-7020, 2010.

P.M.Gruber
Asymptotic estimates for best and stepwise approximation of convex bodies.i
Forum Math. 5(5):281-297, 1993.

P.M.Gruber
Convex and discrete geometry
Springer, Berlin. 2007.

E. Hlawka
Ausfüllung und Überdeckung konvexer Körper durch konvexe Körper

Monatsh. Math. 53(1):81-131, 1949.

- C. A. Rogers
  Packing and covering
  Cambridge University Press. 1964.

- A.Aspremont, S.Boyd
  Relaxations and Randomized Methods for Nonconvex QCQPs
  Stanford University. 2003.

- Dower, P.M. and McEneaney, W.M.
  A max-plus based fundamental solution for a class of infinite dimensional Riccati equations
  Proc. of the 2011 CDC :615 -620, 2011.

- Sridharan, Srinivas and Gu, Mile and James, Matthew R. and McEneaney, William M.

Reduced-complexity numerical method for optimal gate synthesis
Phys. Rev. A 82(4):042319, 2010.

G.Cohen, S.Gaubert and J-P. Quadrat
Kernels,images and projections in dioids
Proc. of WODES, 1996

W.M.McEneaney and J.Kluberg
Convergence Rate for a Curse-of-Dimensionality-Free Method for a Class of HJB PDEs
SIAM J. Control Optim., 48(5),3052-3079,2009.

S.Gaubert, Z.Qu
The contraction rate in Thompson metric of order-preserving flows on a cone - application to generalized Riccati equations.
accepted pending minor revisions, 2012

Z.Qu
Contraction of Riccati flows applied to the convergence analysis of a max-plus curse of dimensionality free method
submitted to SIAM, 2012

Z.Qu
Numerical methods for Hamilton-Jacobi equations and nonlinear Perron-Frobenius theory
PhD. thesis, 2013