Programmation linéaire colorée

Aspects algorithmiques et combinatoires

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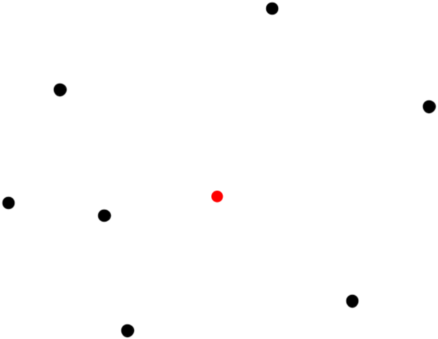
Plan

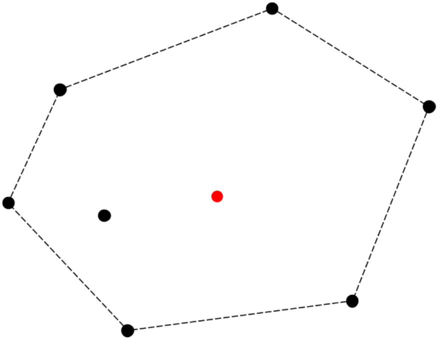
1. Colorful linear programming
2. Some algorithms
3. Counting questions
    - A geometrical problem : the colorful simplicial depth
    - A combinatorial approach : Octahedral systems

Colorful linear programming

The Carathéodory Theorem in dimension two



The Carathéodory Theorem in dimension two



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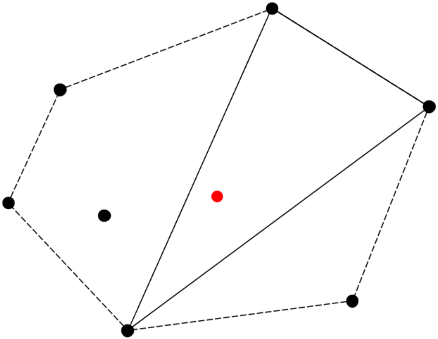


Linear programming

The linear programming problem.

Input : a set $S \subset \mathbb{Q}^d$, a point $p \in \mathbb{Q}^d$.
Output : **Decide** whether there is

$$T \subseteq S, |T| \leq d+1, \text{ such that } p \in \text{conv}(T).$$

If "yes", **find** it.

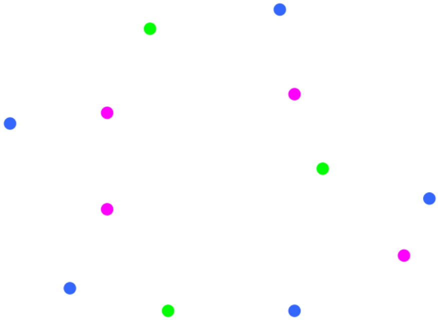
Carathéodory Theorem $\Longrightarrow$ If $p \in \text{conv}(S)$, there is such a T.

Complexity status

Theorem

The functional problem of linear programming is in $\mathcal{P}$.

The colorful Carathéodory Theorem in dimension two



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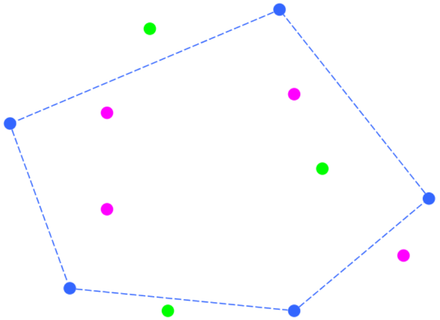

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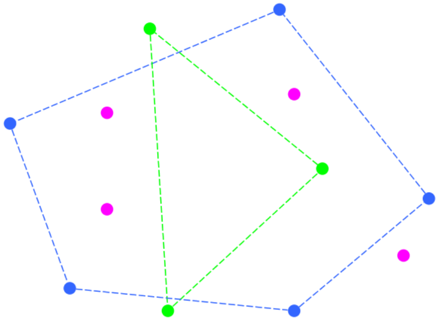

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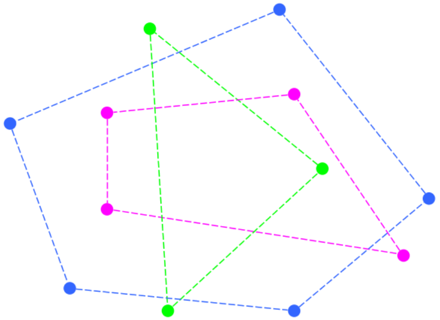

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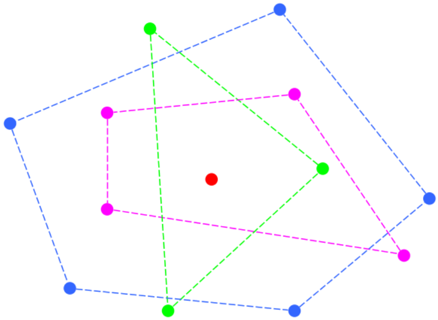

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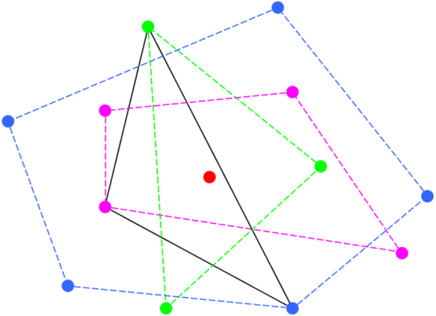


The colorful Carathéodory Theorem [Bárány 1982]

Theorem

Let $S_1, \ldots, S_{d+1}$ be sets of points, and a point $p \in \mathbb{R}^d$. If $p \in \bigcap_{i=1}^{d+1} \mathrm{conv}(S_i)$ (Bárány's conditions), there is $T \subseteq \bigcup_{i=1}^{d+1} S_i$ such that

$$|T \cap S_i| \leq 1 \text{ for all } i \text{ and } p \in \mathrm{conv}(T).$$

$T \subseteq \bigcup_{i=1}^{d+1} S_i$ such that $|T \cap S_i| \leq 1$ for $i = 1, \ldots, d+1$ is **colorful.**

Colorful linear programming [Bárány and Onn in 1997]

The colorful linear programming problem.

Input : k sets, or *colors*, $S_1, \ldots, S_k \subset \mathbb{Q}^d$, a point $p \in \mathbb{Q}^d$.
Output : **Decide** whether there is

a colorful T such that $p \in \text{conv}(T)$.

If "yes", **find** it.

Colorful Carathéodory Theorem $\Longrightarrow$ If $k = d + 1$ and $p \in \bigcap_{i=1}^{k} \text{conv}(S_i)$, there is such a T.

Complexity status

Theorem (Bárány and Onn, 1997)

The decision problem of colorful linear programming is $\mathcal{NP}$-complete.

Upcoming result (Meunier and Sarrabezolles) : It is harder than a $\mathcal{PPAD}$-complete problem.

Other colorful linear programming problems

Problem under Bárány's conditions.

Input : $d+1$ sets $S_1, \ldots, S_{d+1} \subset \mathbb{Q}^d$ and a point $p \in \mathbb{Q}^d$ such that, $p \in \bigcap_{i=1}^{d+1} \operatorname{conv} S_i$.
Output : Find a colorful simplex containing p.

Complexity : open question.

Other colorful linear programming problems

Lemma (Octahedral Lemma)

Let $X_1, \ldots, X_{d+1}$ be sets of points, with $|X_i| = 2$, and a point p. There is an even *number of colorful simplices generated by $\bigcup_{i=1}^{d+1} X_i$ containing p.*

In particular, if there is one there is another.

Other colorful linear programming problems

Another colorful simplex.

Input : A colorful simplex σ containing *p* and a colorful simplex σ' disjoint from σ.
Output : Find another colorful simplex containing *p* generated by points of $\sigma \cup \sigma'$.

Complexity : This problem is $\mathcal{PPAD}$-complete (Meunier and Sarrabezolles).

Some algorithms [1]

[1] In this section, p is the origin $\mathbf{0}$ and the points are in general position.

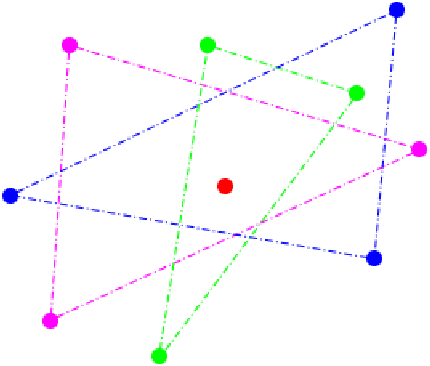
Problem under Bárány's conditions.

Input : $d+1$ sets $S_1, \ldots, S_{d+1} \subset \mathbb{Q}^d$ and a point $p \in \mathbb{Q}^d$ such that, $p \in \bigcap_{i=1}^{d+1} \operatorname{conv} S_i$.
Output : Find a colorful simplex containing p.

Complexity : open question.

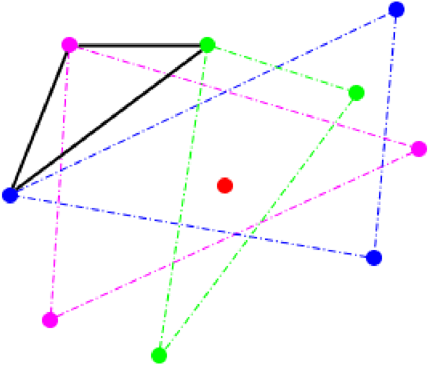
Bárány's algorithm (1982)

Consider $S_1, \ldots, S_{d+1}$, sets of points, each containing **0**.



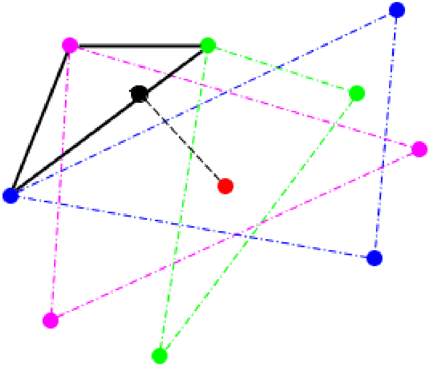
Bárány's algorithm (1982)

Consider a colorful simplex.



Bárány's algorithm (1982)

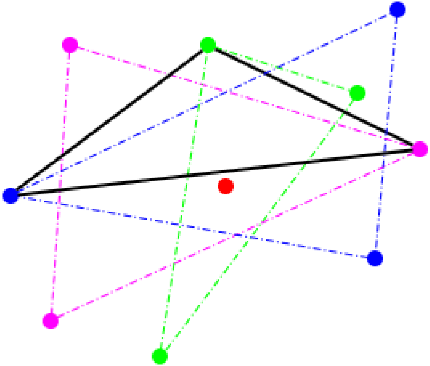
Consider the closest point to the **0** in this simplex.



This point lies on a facet of the colorful simplex. A color *i* is missing on this facet.

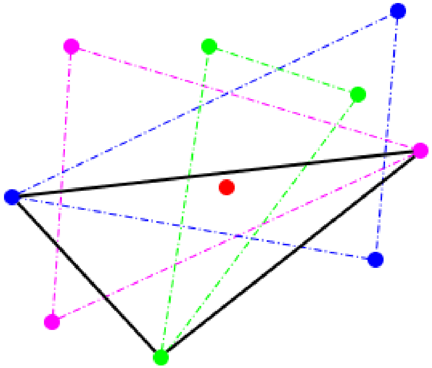
Bárány's algorithm (1982)

Replace the vertex of color *i* with another vertex of the same color, getting a point closer to **0**



Bárány's algorithm (1982)

Iterate...



Bárány's algorithm

Complexity for rational data :
Given $\rho > 0$ and $S_1, \ldots, S_{d+1} \subset \mathbb{Q}^d$ of bit size L, with $B(0, \rho) \subset \text{conv}(S_i)$.

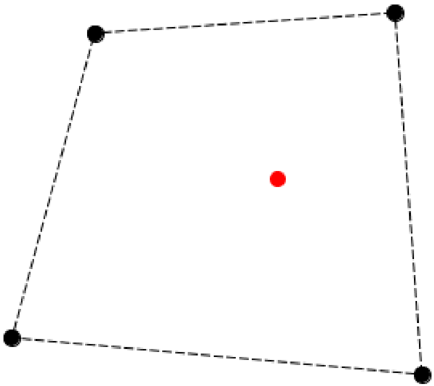
This algorithm find a colorful simplex containing $\mathbf{0}$ in polynomial time in L and $1/\rho$.

Another colorful simplex.

Input : A colorful simplex σ containing p and a colorful simplex σ' disjoint from σ.
Output : Find another colorful simplex containing p generated by points of $\sigma \cup \sigma'$.

Complexity : This problem is $\mathcal{PPAD}$-complete (Meunier and Sarrabezolles).

A point in the convex hull of $d+2$ points in $\mathbb{R}^d$ is in exactly two simplices generated by those points.



Reminder : the simplex algorithm

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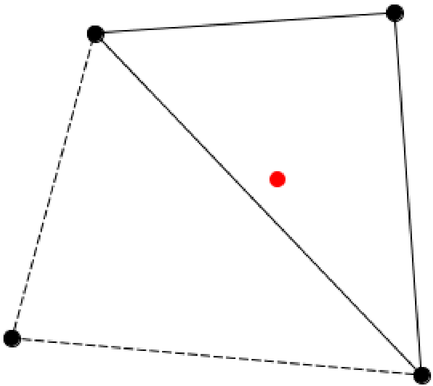

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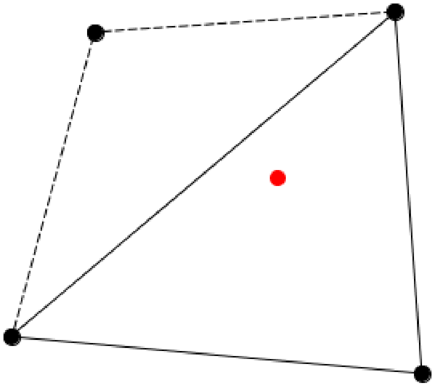
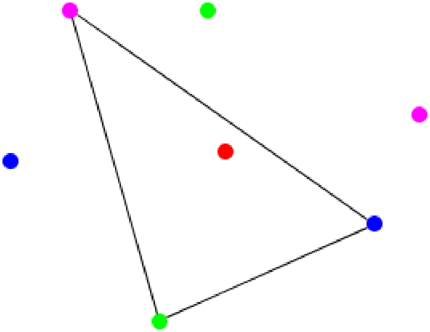


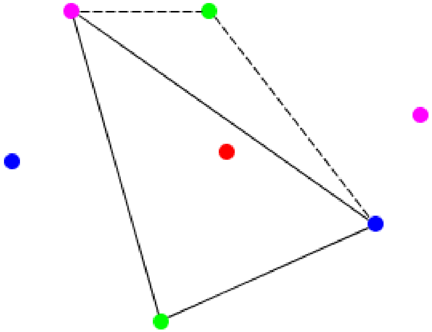
Pivoting algorithm

Consider a colorful simplex σ containing **0**, and a disjoint colorful simplex *i.e. one point of each color not in* σ.

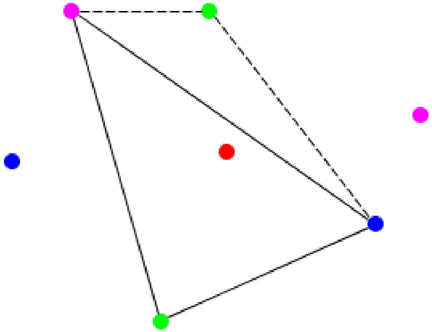


Pivoting algorithm

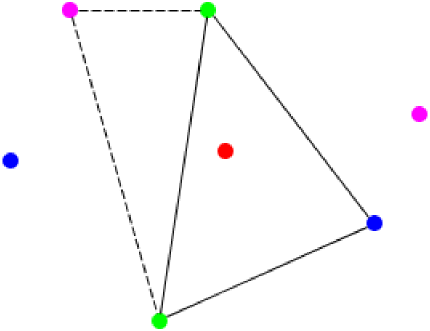
Consider a color, called the pivoting color.



Apply the argument of the simplex algorithm.

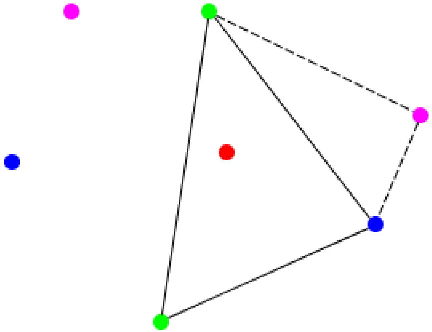


Consider the other simplex containing the origin.

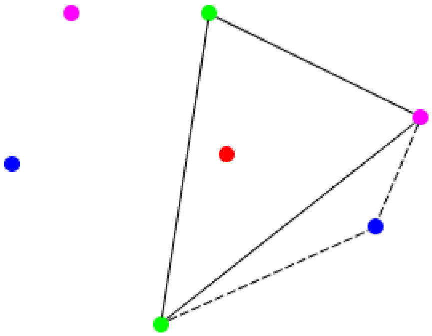


This simplex is "almost" colorful. The pivoting color is duplicated, and a color i is missing.

Add the vertex of color i not in σ, and obtain a new simplex containing **0**.



Iterate...



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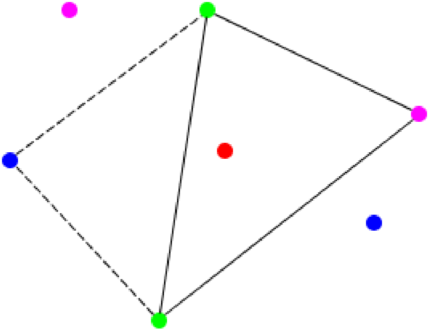

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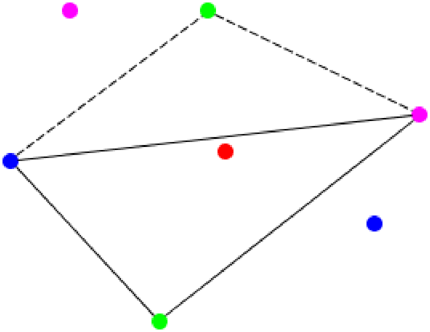
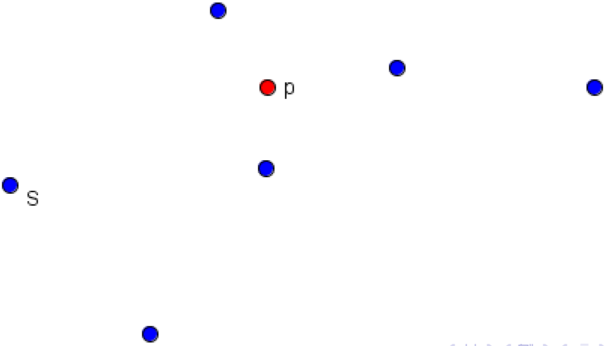


Counting feasible bases

Let S be a set of points in $\mathbb{R}^d$.

Simplicial depth of a point p = number of d-simplices generated by S and containing p.



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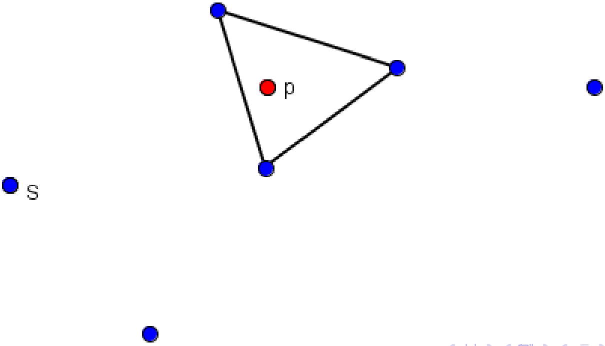

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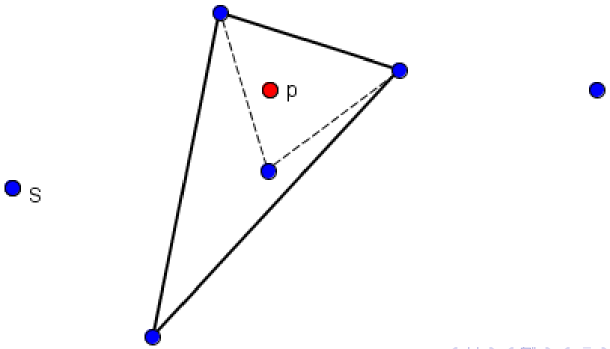

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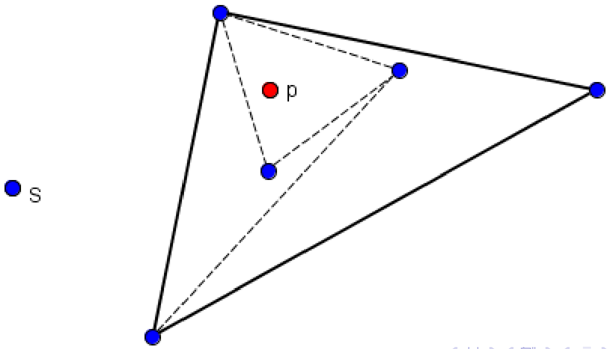

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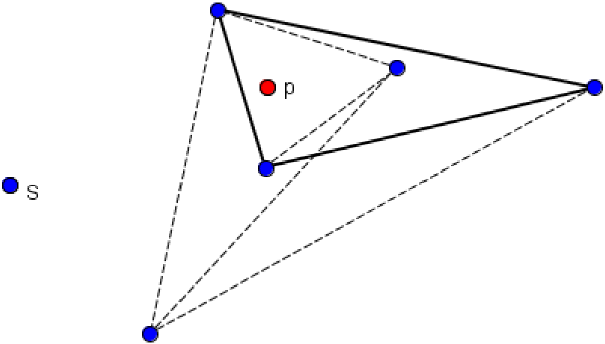

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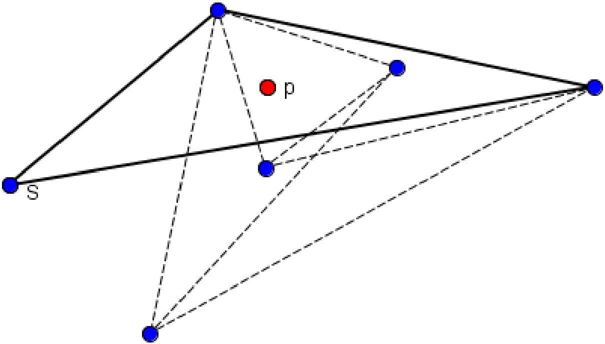

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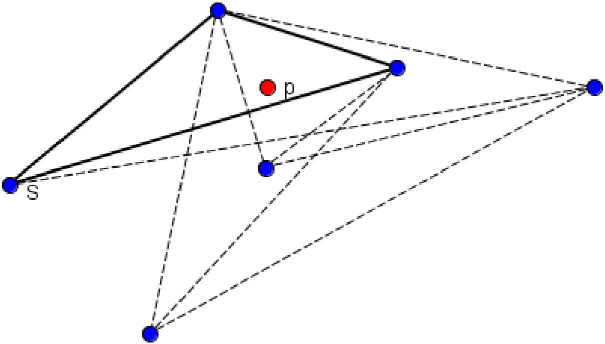
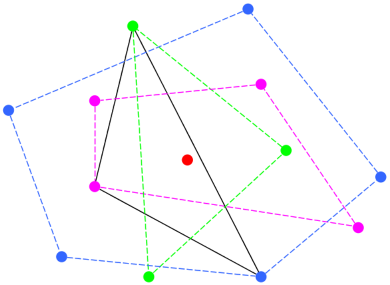


Original motivation : simplicial depth

Let $S_1, \ldots, S_{d+1}$ be $(d+1)$ sets of points in $\mathbb{R}^d$.

Colourful simplicial depth of a point p is :

depth$_{S_1,\ldots,S_{d+1}}(p)$ = number of colourful d-simplices generated by $\bigcup_{i=1}^{d+1} S_i$ and containing p.



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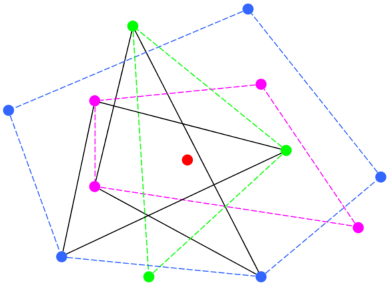

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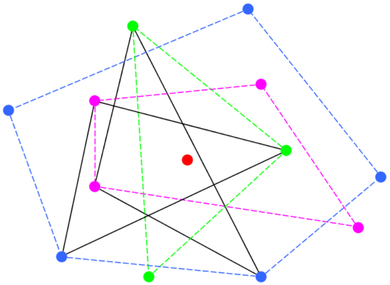


$$\mu(d) = \min_{S_1,\ldots,S_{d+1},p} \mathbf{depth}_{S_1,\ldots,S_{d+1}}(p).$$

A lower bound on simplicial depth

For $S \cup \{p\}$ in general position

[Bárány1982]

$$\max_p \operatorname{depth}_S(p) \geq \frac{1}{(d+1)^{d+1}} \binom{n}{d+1} \quad \text{with } n = |S|.$$

Proof combines the Tverberg theorem and the colourful Carathéodory theorem.

A lower bound on simplicial depth

For $S \cup \{p\}$ in general position

[Bárány1982]

$$\max_p \operatorname{depth}_S(p) \geq \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} \quad \text{with } n = |S|.$$

Proof combines the Tverberg theorem and the colourful Carathéodory theorem.

A new lower bound for simplicial depth

$$\mu(d) = \min_{\substack{S_1,\ldots,S_{d+1} \\ p \in \bigcap_{i=1}^{d+1} \text{conv}(S_i)}} \#\{T : T \text{ colourful and } p \in \text{conv}(T)\}.$$

Strong version of Colourful Carathéodory Theorem : each point in $\bigcup_{i=1}^{d+1} S_i$ is part of a colourful simplex containing the origin.

$$\max_p \text{depth}_S(p) \geq \frac{\mu(d)}{(d+1)^{(d+1)}} \binom{n}{d+1} \quad \text{with } n = |S|.$$

What is the exact value of $\mu(d)$?

A new lower bound for simplicial depth

$$\mu(d) = \min_{\substack{S_1,\ldots,S_{d+1} \\ p \in \bigcap_{i=1}^{d+1} \mathrm{conv}(S_i)}} \#\{T : T \text{ colourful and } p \in \mathrm{conv}(T)\}.$$

Strong version of Colourful Carathéodory Theorem : each point in $\bigcup_{i=1}^{d+1} S_i$ is part of a colourful simplex containing the origin.

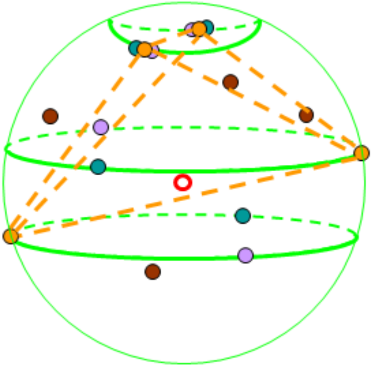
$$\max_p \mathrm{depth}_S(p) \geq \frac{d+1}{(d+1)^{(d+1)}} \binom{n}{d+1} \quad \text{with } n = |S|.$$

What is the exact value of $\mu(d)$?

Upper bound on the colourful simplicial depth

Unfortunately,
[Deza et al., 2006]

$$\mu(d) \leq d^2 + 1.$$



Gromov's bound

$$\max_p \text{depth}_S(p) \geq \frac{\mu(d)}{(d+1)^{(d+1)}} \binom{n}{d+1} \quad \text{with } n = |S|,$$

with $\mu(d) = d^2 + 1$ at best.

[Gromov, 2010]

$$\max_p \text{depth}_S(p) \geq \frac{2d}{(d+1)!(d+1)} \binom{n}{d+1} \quad \text{with } n = |S|.$$

(simplification by Karasev, 2012).

The conjecture

Conjecture.

$$\mu(d) = d^2 + 1.$$

The successive improvements

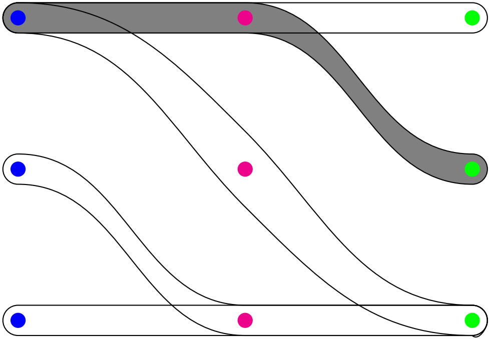
	Lower bound for $\mu(d)$	Conjecture true for d up to
Bárány, 1982	$d+1$	1
Deza et al., 2006	$2d$	2
Bárány and Matoušek, 2007	$\max(3d, \frac{1}{5}d^2+\frac{1}{5}d)$	3
Stephen and Thomas, 2008	$\frac{1}{4}d^2+d+1$	$\emptyset$
Deza, Stephen, and Xie, 2011	$\frac{1}{2}d^2+d+\frac{1}{2}$	$\emptyset$
Deza, Meunier, and S., 2012	$\frac{1}{2}d^2+\frac{7}{2}d-8$	4

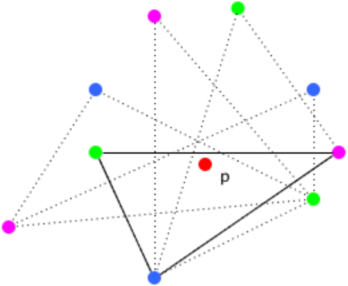
A combinatorial counterpart : octahedral systems

An *octahedral system* Ω in an n-partite hypergraph $(V_1, \ldots, V_n, E)$ satisfying *parity condition :* for any $X \subseteq \bigcup_{i=1}^{n} V_i$ such that $|X \cap V_i| = 2$ for all i, the number of edges of Ω induced by X is even.

Octahedral systems *without isolated vertex* generalize colorful configurations.

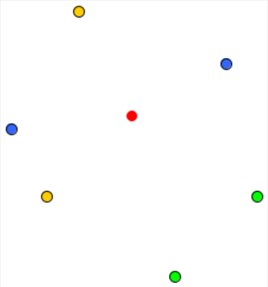
An octahedral system





Two main properties for the geometrical approach

Octahedral Lemma



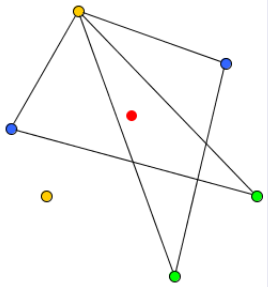
$X \subseteq S$, $|X \cap S_i| = 2$ for all $i \longrightarrow$ an even number of colorful simplices.

Strong colorful Carathéodory Theorem

If $\mathbf{0} \in \text{conv}(S_i)$ for all i, each point is part of some colorful simplices containing the origin.

Two main properties for the geometrical approach

Octahedral Lemma



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Strong colorful Carathéodory Theorem

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Combinatorial approach

Vertex set :
$V = V_1 \cup \cdots \cup V_{d+1}$.

Edge set : E.

Parity condition : The number of edges induced by X, with $|X \cap V_i| = 2$ for all i, is even.

Octahedral systems without isolated vertex : Every point in $\bigcup_{i=1}^{d+1} V_i$ is in at least one edge.

Geometrical approach

A colorful configuration
$S = S_1 \cup \cdots \cup S_{d+1}$.

colorful simplices containing the origin.

Octahedral Lemma : The number of colorful simplices containing the origin generated by points in X, with $|X \cap S_i| = 2$ for all i, is even.

Strong colorful Carathéodory Theorem : Every point in $\bigcup_{i=1}^{d+1} S$ is part of some colorful simplex containing the origin.

If Ω realizes a colorful configuration, **the number of edges $|E|$ is the number of colorful simplices containing the origin.**

Definition (ν)

$\nu(d)$ is the minimal number of edges of an octahedral system without isolated vertex with $|V_i| = d + 1$ for $i = 1, \ldots, d + 1$.

$$\nu(d) \leq \mu(d)$$

Lower bounds

Theorem (Deza, Meunier, S.)

$$\nu(d) \geq \frac{1}{2}d^2 + \frac{7}{2}d - 8$$

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Idea of the proof : induction

Inductive approach.

Given an octahedral system $\Omega = (V, E)$ without isolated vertex and one of its vertices v, use the bound for $\Omega' = (V', E') = \Omega \setminus \{v\}$:

$$|E| = |E'| + \deg_\Omega(v).$$

For any such Ω', parity condition automatically satisfied.

We would like to ensure that Ω' is again without isolated vertex.

Main Idea. Delete the vertices one after another until reaching an octahedral system whose number of edges can be estimated.

Small instances

An octahedral system with $n = 5$, $|V_1| = \ldots = |V_5| = 5$ *and without isolated vertex has at least* 17 *edges.*

Proposition

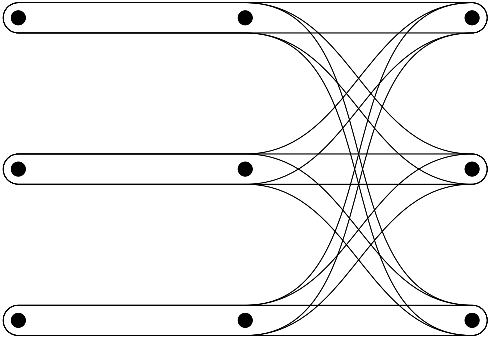
$$\mu(4) = 17.$$

Computational approach "branch-and-bound" $\mu(4) \geq 14$, (Deza, Stephen, and Xie, 2012).

Realisability

Is any octahedral system Ω with $|V_i| = d + 1$ for $i = 1, \ldots, d + 1$ and without isolated vertex the combinatorial counterpart of sets of points $S_1, \ldots, S_{d+1}$ in $\mathbb{R}^d$?

No. Counterexample.



Consequence

It might be possible that the conjecture $\mu(d) = d^2 + 1$ cannot be proven using octahedral systems...

Open questions

- Complexity status of the feasibility problem under the condition $p \in \bigcap_{i=1}^{d+1} \text{conv}(S_i)$.
- $\mu(d) \stackrel{?}{=} d^2 + 1$.

Thank you.