Proof of Myerson's Lemma

① Implementable $\Rightarrow$ monotone.

$\forall\, v_i, v_i', b_{-i}$, DSIC $\Rightarrow$

$$\left.\begin{aligned} x_i(v_i, b_{-i})\cdot v_i - p_i(v_i, b_{-i}) &\geq x_i(v_i', b_{-i})\cdot v_i - p_i(v_i', b_{-i}) \\ x_i(v_i', b_{-i})\cdot v_i' - p_i(v_i', b_{-i}) &\geq x_i(v_i, b_{-i})\cdot v_i' - p_i(v_i, b_{-i}) \end{aligned}\right\} \overset{+}{\Rightarrow}$$

$$\left(x_i(v_i, b_{-i}) - x_i(v_i', b_{-i})\right)\cdot\left(v_i - v_i'\right) \geq 0$$

Hence, for all b_{-i}, $x_i(\cdot, b_{-i})$ is non-decreasing.

② Implementable $\Rightarrow$ payment is essentially unique

Fix i, b_{-i}: $\quad u_i(v_i, b_{-i}) = x_i(v_i, b_{-i})\cdot v_i - p_i(v_i, b_{-i})$

$$\forall\, v, \varepsilon: \left.\begin{aligned} u_i(v_i+\varepsilon, b_{-i}) &\geq x_i(v_i, b_{-i})(v_i+\varepsilon) - p_i(v_i, b_{-i}) \\ u_i(v_i, b_{-i}) &\geq x_i(v_i+\varepsilon, b_{-i})\cdot v_i - p_i(v_i+\varepsilon, b_{-i}) \end{aligned}\right\} \Leftrightarrow$$

$$\left.\begin{aligned} u_i(v_i+\varepsilon, b_{-i}) - u_i(v_i, b_{-i}) &\geq x_i(v_i, b_{-i}) \cdot \varepsilon \\ u_i(v_i+\varepsilon, b_{-i}) - u_i(v_i, b_{-i}) &\leq x_i(v_i+\varepsilon, b_{-i}) \cdot \varepsilon \end{aligned}\right\} \Rightarrow$$

$$x_i(v_i, b_{-i}) \cdot \varepsilon \leq u_i(v_i+\varepsilon, b_{-i}) - u_i(v_i, b_{-i}) \leq x_i(v_i+\varepsilon, b_{-i}) \cdot \varepsilon \quad (**)$$

x : implementable $\Rightarrow x_i(\cdot, b_{-i})$ non-decreasing

$\Rightarrow x_i$: Rieman integrable

$$\overset{(**)}{\Rightarrow} u_i(z, b_{-i}) - u_i(0, b_{-i}) = \int_0^z x_i(t, b_{-i})\, dt$$

$$\Rightarrow p_i(z, b_{-i}) = x_i(z, b_{-i}) \cdot z - \int_0^z x_i(t, b_{-i})\, dt + p_i(0, b_{-i}) \quad (*)$$

3 Implementable, NPT, IR $\Rightarrow$ payment is unique.

$$\left.\begin{aligned} NPT &\Rightarrow p_i(0, b_{-i}) \geq 0 \quad , \forall\, b_{-i} \\ IR &\Rightarrow p_i(0, b_{-i}) \leq 0 \quad , \forall\, b_{-i} \end{aligned}\right\} \Rightarrow p_i(0, b_{-i}) = 0, \forall\, b_{-i}$$

4 monotone $\Rightarrow$ implementable

suppose $x_i(\ ,b_{-i})$ is non-decreasing $\forall i, b_{-i}$

<u>Claim</u>: Combined with payments as in (*), (x,p) is DSIC.

<u>Proof</u>: ∘ fix i, v_i (true type), v_i' (candidate mis-report), b_{-i}:

$$A = x_i(v_i, b_{-i})\cdot v_i - p_i(v_i, b_{-i}) = \int_0^{v_i} x_i(t, b_{-i})\,dt$$

$$B = x_i(v_i', b_{-i})\cdot v_i - p_i(v_i', b_{-i}) =$$

$$= x_i(v_i', b_{-i})\cdot(v_i - v_i') + \int_0^{v_i'} x_i(t, b_{-i})\,dt$$

$$= x_i(v_i', b_{-i})\cdot(v_i - v_i') + \int_{v_i}^{v_i'} x_i(t, b_{-i})\,dt + \int_0^{v_i} x_i(t, b_{-i})\,dt$$

$$\leq x_i(v_i', b_{-i})\cdot(v_i - v_i') + (v_i' - v_i)\,x_i(v_i', b_{-i}) + \int_0^{v_i} x_i(t, b_{-i})\,dt$$

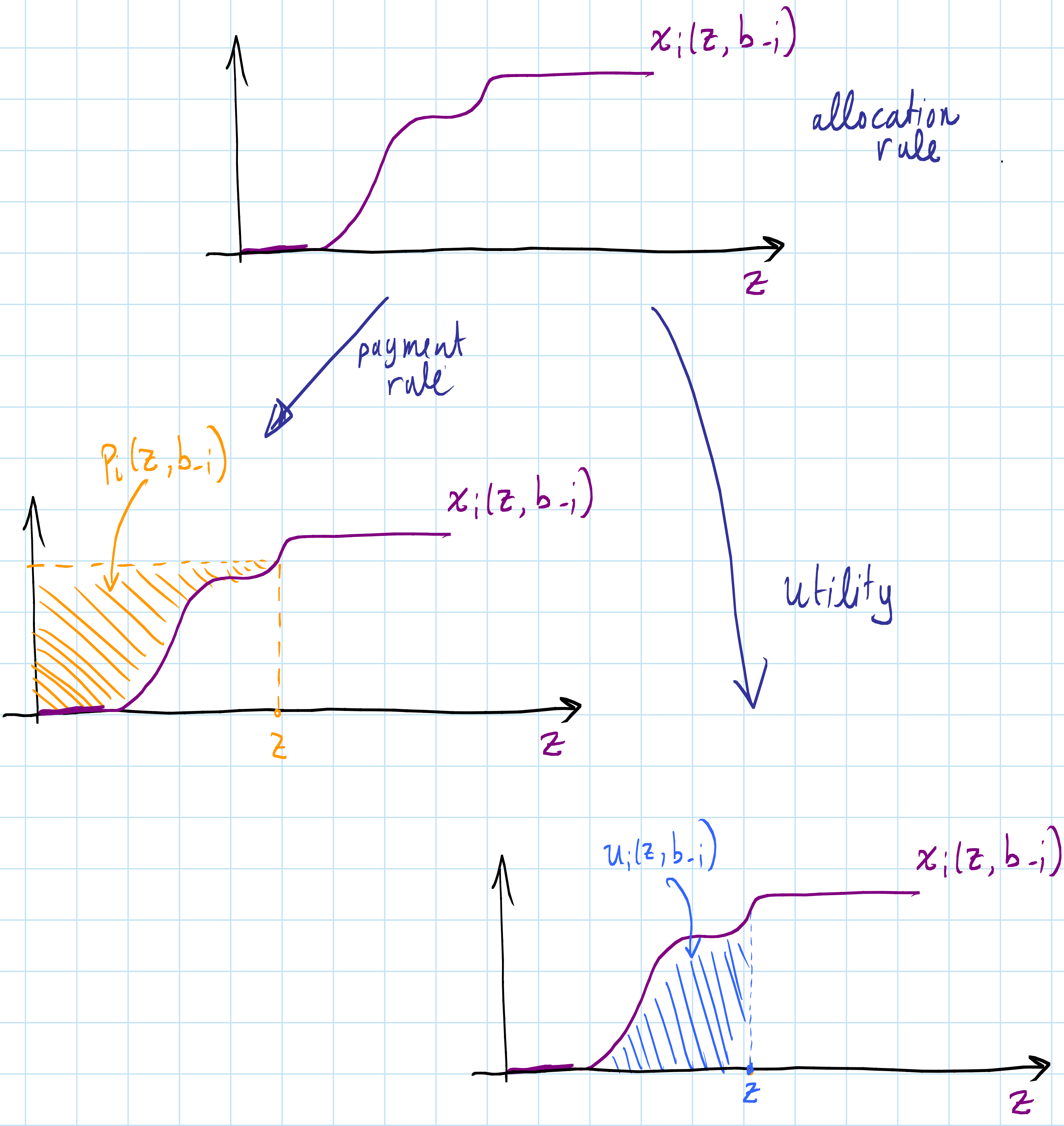
$x_i(\cdot, b_{-i})$ non-decreasing

$$\leq A.$$

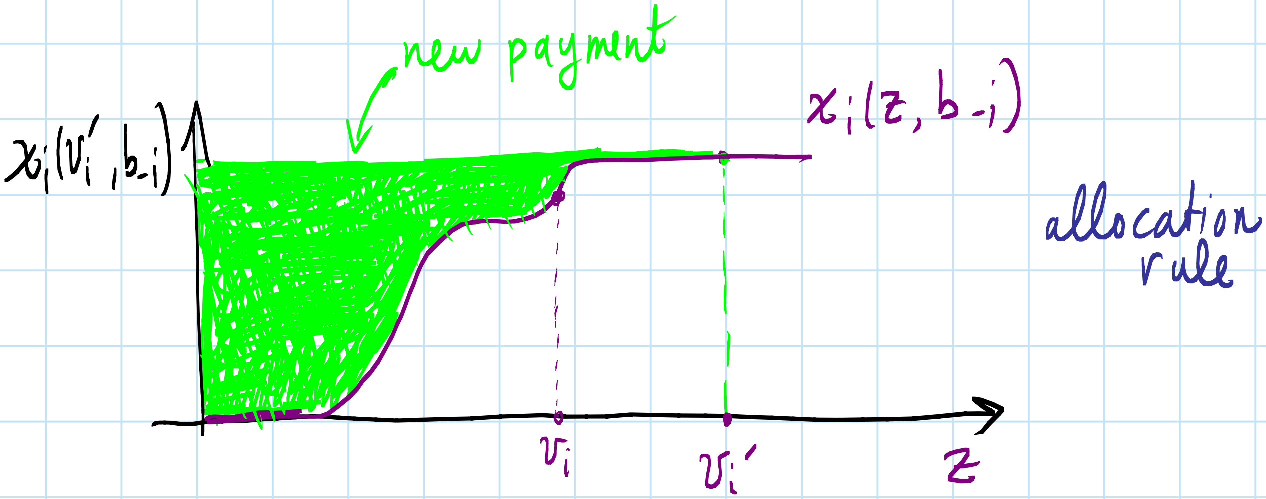
☒

· **Illustration:**

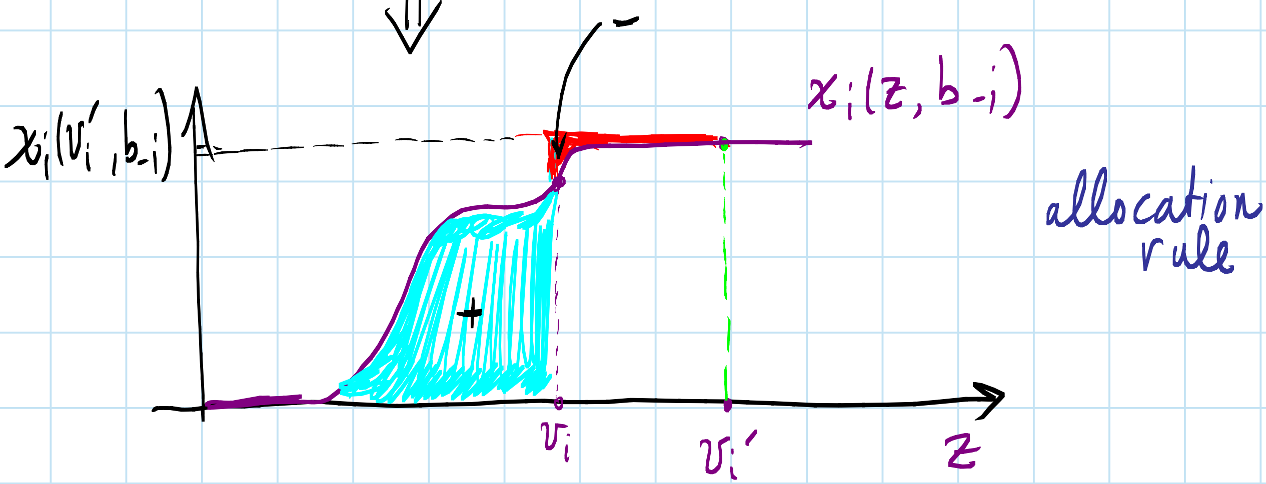
fix i, b_{-i}: allocation to bidder i must be monotone in his bid



• If true type is v_i, then misreporting to $v_i' \geq v_i$ results in

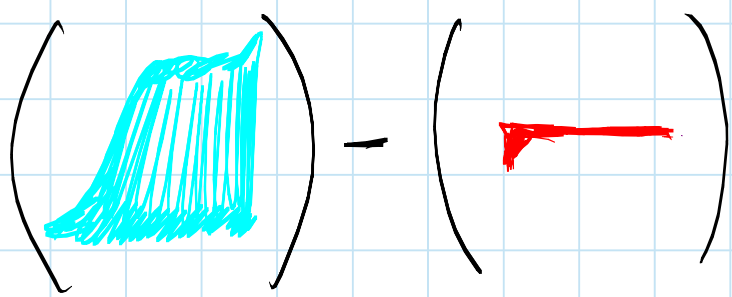


$\Downarrow$



new utility:

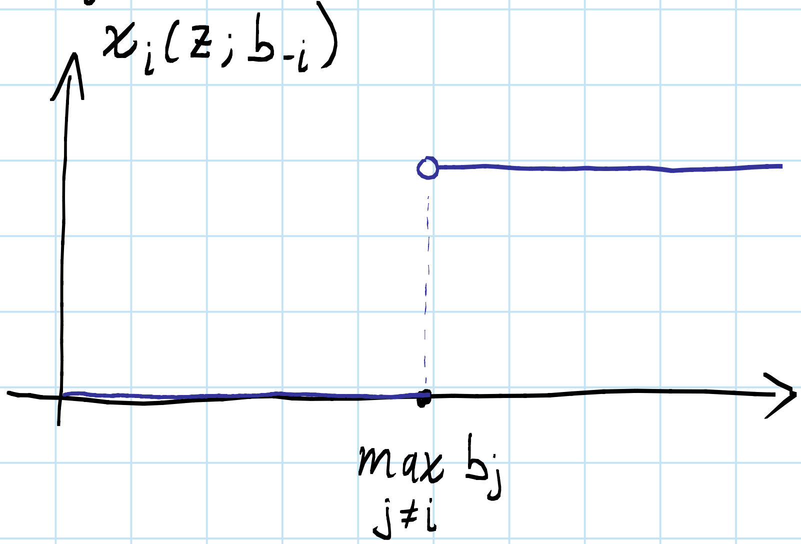
$u_i(v_i', b_{-i}) =$



$= u_i(v_i, b_{-i})$

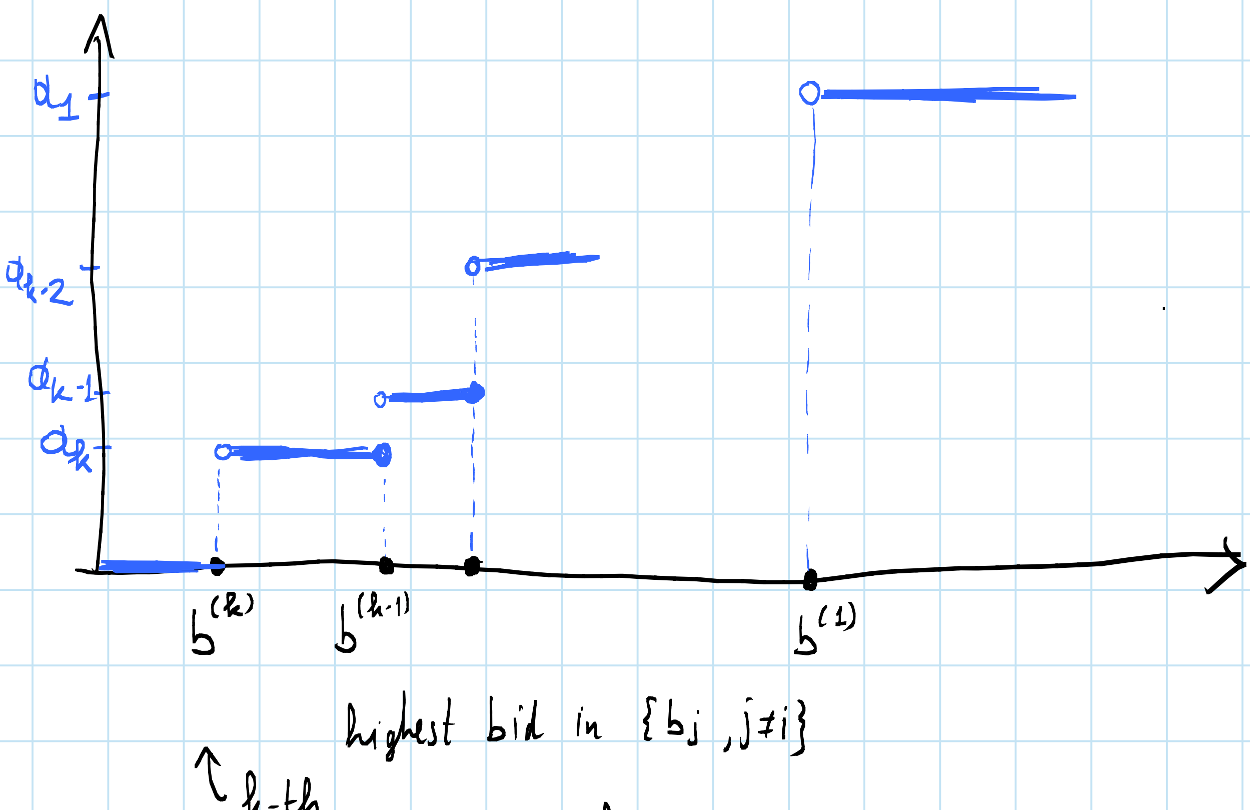
- Applications of Myerson's Lemma:

- Single-Item Auction: Allocate to highest bidder



$$p_i(z; b_{-i}) = \begin{cases} \max_{j \neq i} b_j & , \; z > \max_{j \neq i} b_j \\ 0 & , \; \text{o.w.} \end{cases}$$

- Sponsored Search: Use Greedy Algorithm



$$p_i(z; b_{-i}) = \sum_{\ell=1}^{k} (\alpha_\ell - \alpha_{\ell+1}) \cdot b^{(\ell)} \cdot \mathbb{1}_{\{z \geq b^{(\ell)}\}}$$