Polynomial time algorithms for the lot-sizing problem under energy constraints

Ayse Akbalik*, Céline Gicquel#, Bernard Penz◊, Christophe Rapine*

* Université de Lorraine
# Université Paris Saclay, LRI
◊ Université Grenoble Alpes, CNRS, G-SCOP

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Context

- Rarefaction of fossil fuels
  ⇒ Increase in energy price, price fluctuation of the electricity
- Use of less polluting but less flexible renewable energies
  ⇒ Limitation on the available energy
- New environmental standards, quotas, taxes (Kyoto, COP21)
  ⇒ Re-organization of industrial activities (to be more ecological, less energy consuming, etc.)
Scope of the talk

- Single-item lot sizing problem
- Identical, parallel and capacitated machines
- Energy limitation for machine start-ups and production
- What to decide: How many machines to start and a feasible production plan for them
- Reduction to a capacity acquisition problem
Lot-sizing problem on a time horizon \( T \) on parallel machines

- \( d_t \): demand in period \( t \)
- \( c_t, h_t \): unit production cost, unit holding cost in \( t \).
  
  We assume that costs are non-speculative: \( c_t + h_t \geq c_{t+1} \)
- \( U \): capacity of each machine
- \( f \): start-up cost of a machine

With energy limitations

- \( E_t \): amount of available energy in period \( t \)
- \( p_t \): energy consumption to produce one unit in period \( t \)
- \( w_t \): energy consumption to start a machine in period \( t \)
energy-LSP

\[
\begin{align*}
\text{min} & \quad \sum_{t=1}^{T} fm_t^+ + \sum_{t=1}^{T} c_t x_t + \sum_{t=1}^{T} h_t s_t \\
\text{s.t.} & \quad s_{t-1} + x_t = s_t + d_t \quad \forall t \in \{1..T\} \quad (1) \\
& \quad x_t \leq m_t U \quad \forall t \in \{1..T\} \quad (2) \\
& \quad p_t x_t + w_t m_t^+ \leq E_t \quad \forall t \in \{1..T\} \quad (3) \\
& \quad m_t = m_{t-1} + m_t^+ \quad \forall t \in \{1..T\} \quad (4) \\
& \quad s_t \geq 0, x_t \geq 0, m_t \in \mathbb{Z}^+, m_t^+ \in \mathbb{Z}^+ \quad \forall t \in \{1..T\} \quad (5)
\end{align*}
\]

- \(x_t\): quantity produced in \(t\)
- \(s_t\): stock level at the end of period \(t\)
- \(m_t\): the number of machines running during period \(t\)
- \(m_t^+\): the number of machines started-up at the beginning of period \(t\)
energy-LSP with $w_t = 0$

We consider a much simpler case where $w_t = 0$

- Energy is consumed only by production: define a limit $F_t \equiv E_t/p_t$ on the number of units produced in period $t$
- We have to decide how many machines $m$ to start at the beginning of the horizon, and a feasible plan for them

\[
\begin{align*}
\min & \quad fm + \sum_{t=1}^{T} c_t x_t + \sum_{t=1}^{T} h_t s_t \\
\text{s.t.} & \quad s_{t-1} + x_t = s_t + d_t \quad \forall t \in \{1..T\} \quad (1) \\
& \quad x_t \leq C \equiv mU \quad \forall t \in \{1..T\} \quad (2) \\
& \quad x_t \leq F_t \equiv E_t/p_t \quad \forall t \in \{1..T\} \quad (3) \\
& \quad s_t \geq 0, x_t \geq 0, m \in \mathbb{Z}^+ \quad \forall t \in \{1..T\} \quad (5)
\end{align*}
\]
energy-LSP with $w_t = 0$

Capacity acquisition problem ($C = mU$) with additional production constraints ($F_t = E_t/p_t$)
We consider also the case with $p_t = 0$

- Energy is consumed only to start a machine
- Does not limit the production, but the number of machines we can start at a period
- Most probably, we can not start all the machines we wish at the beginning of the horizon!

We can formulate the problem as before, with additional production constraints defined by:

$$F_t = \sum_{u=1}^{t} \lfloor E_u / w_u \rfloor$$

*We want to solve a capacity acquisition lot-sizing problem (with additional capacity constraints)*
There are 3 important domains to position our problem:

- **Single-item lot sizing problem**
  - Wagner and Whitin (1958)
  - Florian et al. (1980)
  - Bitran and Yanasse (1982)
  - Reviews: Brahimi et al. (2006), Gicquel et al. (2008)

- **Energy planning in manufacturing**
  - Artigues et al. (2013)
  - Waldemarsson et al. (2013)
  - Schultz et al. (2015)
  - Review: Gahm et al. (2016)

- **Capacity planning (acquisition)**
  - Review: Martinez-Costa et al. (2014)
Literature

- Masmoudi et al. (2015)
- Atamtürk and Hochbaum (2001)
  Li and Meissner (2011)

Single-item lot sizing problem

- Energy planning in manufacturing
- Capacity planning (acquisition)

Our study
Pure capacity acquisition problem

We consider first the capacity acquisition problem, without additional energy constraints

- Capacity $C = mU$ is considered as continuous

The approach of Atamtürk & Hochbaum (2001):

- For a given capacity $C$, an optimum solution $x$ can be found in linear time $O(T)$
- There are at most $O(T^2)$ capacities to consider for extreme solutions: Overall time complexity in $O(T^3)$

*We improve dramatically their algorithm complexity (but without subcontracting)*
No-Slack solution

We only consider the solutions $x$ such that:

- either $t$ is a full production period, that is, $x_t = C$
- or $t$ is a regeneration period, that is, $s_{t-1} = 0$

In fact, we are mainly interested in the set of periods at full production capacity

$$\mathcal{B} = \{ t | x_t = C \}$$

Idea: forget about solution $x$, only focus on set $\mathcal{B}$!
For a set $B$ of periods:

On the example, $B = \{3, 4, 6\}$
Set $\mathcal{A}$

For a set $\mathcal{B}$ of periods:

Define $\mathcal{A} = \{ t \mid t \notin \mathcal{B} \text{ and } (t + 1) \in \mathcal{B} \}$

-demands 10 12 6 20 8 14
Associated solution $x(\mathcal{B}, C)$

For a set $\mathcal{B}$ of periods and a capacity $C$

Define the no-slack solution $x_t(\mathcal{B}, C) =$

- Capacity $C$ if $t \in \mathcal{B}$
- Demand $d_t$ if $t \notin \mathcal{A} \cup \mathcal{B}$
- What is needed if $t \in \mathcal{A}$
What is needed?

Write flow conservation till the next regeneration point (between periods $t$ and $v(t)$):

$$x_t(B, C) = D_{t,v(t)} - (v(t) - t)C$$

where $D_{t,v(t)} = \sum_{k=t}^{v(t)} d_k$ is the total demand on $\{t, \ldots, v(t)\}$
Solution $x(\mathcal{B}, C)$ may be unfeasible: $x_t > C$ in some $\mathcal{A}$-period.

On the example, with $\mathcal{B} = \{3, 4, 6\}$ and $C = 12$, production in period 2 exceeds $C$.

What is needed may exceed what is possible
Optimality

Property
If solution $x(B, C)$ is feasible, and produces exactly the total demand ($s_T = 0$), then it is an optimal solution for the problem with capacity $C$.

If $x(B, C)$ is dominant, its cost is $\text{OPT}(C)$.

Property
If for all capacities $C$ in an interval $[a, b]$ the solution $x(B, C)$ is dominant, then $\text{OPT}(C)$ varies linearly over $[a, b]$. 
What does optimal value look like?

Piece-wise affine function
What does optimal value look like?

The optimum $\text{OPT} = \min\{\text{OPT}(C) \mid C \geq 0\}$ is reached at one breakpoint.

We want to construct this cost function
Idea of the algorithm

- \( \text{OPT}(C) \) is a piece-wise affine function
- We establish that it admits at most \( T \) breakpoints
- The optimal is reached at one of the breakpoints
- For a given value \( C \), the optimal solution can be constructed in linear time \( O(T) \)

**We find iteratively the breakpoints** \( C_1, C_2, \ldots, C_k : \)

- From a breakpoint \( C_i \) and a set \( B_i \).
- We search the lowest capacity \( C \leq C_i \) for which solution \( x(B_i, C_i) \) is feasible
- This value defines a new breakpoint \( C_{i+1} \), and a new set \( B_{i+1} \)

As \( B_i \subset B_{i+1} \), we have at most \( T \) iterations
First step

Initialize the algorithm with

- \( C_1 = \max_t \{d_t\} \) and \( B_1 = \{t \mid d_t = C_t\} \)
- Clearly \( x(B_1, C_1) \) is feasible (hence optimal for \( C_1 \))
First step

Initialize the algorithm with

- $C_1 = \max_t \{d_t\}$ and $B_1 = \{t|d_t = C_t\}$
- Clearly $x(B_1, C_1)$ is feasible (hence optimal for $C_1$)
What if we decrease $C$?

For $C = 16$, solution $x(B_1, C_1)$ remains feasible.

How much can we decrease $C$?
First case

C may hit a demand, here at period 6

- $C_2 = 14$
- $\mathcal{B}_2 = \mathcal{B}_1 \cup \{6\}$
Second case

A period of set $\mathcal{A}$ may become saturated, here period 3

- $C_3 = 13$
- $\mathcal{B}_3 = \mathcal{B}_2 \cup \{3\}$

What is the next capacity to consider?
Next iteration

$C = 12, 66$ : period 2 (in set $A$) becomes saturated

- $C_4 = 12.66$
- $B_4 = B_3 \cup \{2\}$

When do we stop?
Stopping criteria

If the first period enters set $\mathcal{B}$

- $C_5 = 12$
- $\mathcal{B}_5 = \mathcal{B}_4 \cup \{1\}$

no feasible solution exists for a lower capacity
Stopping criteria

If the first period enters set $\mathcal{B}$ for capacity $C$ (here $C = 12$), no feasible solution exists for a lower capacity

- A necessary condition to be feasible for a capacity $C'$ is that
  $$D_{1,t} \leq tC'$$
  for all periods $t$

- We have $D_{1,v} = vC$, with here $v = 4$
Complexity of the algorithm

- There are at most $O(T)$ breakpoints: (12, 12.66, 13, 14, 20) on the example
- Each breakpoint can be found and evaluated in linear time

Hence we can build function $OPT(C)$ in time complexity $O(T^2)$
The maximum value can be found in time $O(T \log T)$.

Instead of building the optimal planning to evaluate a breakpoint, we compute iteratively the slope of each segment, keeping track of the block structure.
With energy constraints?

- The algorithm is essentially the same: formulas only get more involved to determine the next capacity breakpoint.
- Some energy limitations $F_t$ may also become a breakpoint.
- We have now at most $2T$ breakpoints.

**Theorem**

The optimum capacity value can still be found in time $O(T \log T)$ with energy constraints (that is, if either $w_t = 0$ or $p_t = 0$).
Conclusion

- The algorithm still works if start-up cost $f()$ is a piecewise concave/convex function.
- Perspective: how to solve the problem with both energy consumption on production AND start-up? Far more challenging, since one has to arbitrate in each period how to use the available amount of energy: either to anticipate the demand (produce to stock) or to increase the capacity of the system (start new machines).
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