Expected Revenue = Expected Virtual Welfare

Assumptions (x,p) is BIC interim NPT

\[
\mathbb{E}_{v \sim F} \left[ \sum_i \hat{p}_i(v) \right] = \sum_i \mathbb{E}_{v \sim F} \left[ \hat{p}_i(v) \right]
\]

Myerson Lemma

\[
= \sum_i \mathbb{E}_{v_i \sim F_i} \left[ v_i \cdot \hat{\lambda}_i(v_i) - \int_0^{v_i} \hat{\lambda}_i(t) \, dt \right]
\]

interim allocation to bidder i

\[
\hat{x}_i(v_i) = \mathbb{E}_{v \sim F} \left[ x_i(v, u_i) \right]
\]

interim payment

\[
\hat{\lambda}_i(v_i) = \mathbb{E}_{v \sim F} \left[ \lambda_i(v, u_i) \right]
\]

\[
= \sum_i \mathbb{E}_{v_i \sim F_i} \left[ v_i \cdot \hat{\lambda}_i(v_i) \right] - \sum_i \lim_{v_i \to 0} \int_{v_i = 0}^{t = 0} \hat{\lambda}_i(t) \cdot f_i(v_i) \, dt \, dv_i
\]

\[
= \sum_i \mathbb{E}_{v_i \sim F_i} \left[ v_i \cdot \hat{\lambda}_i(v_i) \right] - \sum_i \int_{v_i = 0}^{+\infty} \hat{\lambda}_i(t) \cdot (1 - F_i(t)) \, dt
\]

\[
= \sum_i \int_{v_i = 0}^{+\infty} v_i \cdot \hat{\lambda}_i(v_i) \cdot f_i(v_i) \, dv_i - \sum_i \int_{v_i = 0}^{+\infty} \hat{\lambda}_i(v_i) \cdot (1 - F_i(v_i)) \, dv_i
\]
= \sum_i \int_{u_i=0}^{+\infty} \hat{\chi}_i(u_i) \cdot \left( u_i - \frac{1-F_i(u_i)}{f(u_i)} \right) f(u_i) \, du_i

= \sum_i \left[ \mathbb{E}_{u_i} \left[ \hat{\chi}_i(u_i) \cdot \phi_i(u_i) \right] \right] = \left[ \mathbb{E}_{v \sim F} \left[ \sum_i \chi_i(v) \cdot \phi_i(v) \right] \right] \Box