Expected Revenue = Expected Virtual Welfare

[Assumptions (x,p) is BIC interim IR NPT]

$$\mathbb{E}_{v\sim F}\left[\sum_i p_i(v)\right] = \sum_i \mathbb{E}_{v\sim F}\left[p_i(v)\right]$$

interim payment
$\hat{p}_i(v_i) \triangleq \mathbb{E}_{v_{-i}}\left[p_i(v_i, v_{-i})\right]$

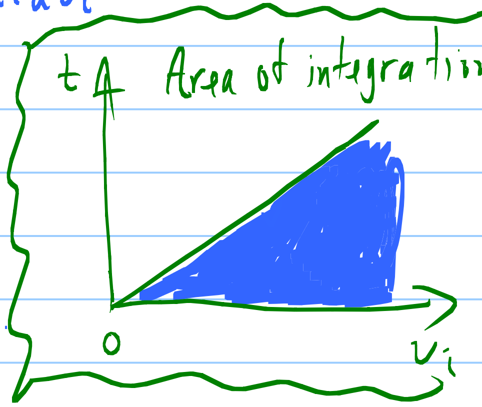
$$= \sum_i \mathbb{E}_{v_i\sim F}\left[\hat{p}_i(v_i)\right]$$

Myerson Lemma

$$= \sum_i \mathbb{E}_{v_i\sim F_i}\left[v_i \cdot \hat{x}_i(v_i) - \int_0^{v_i} \hat{x}_i(t)\,dt\right]$$

interim allocation to bidder i
$\hat{x}_i(v_i) \triangleq \mathbb{E}_{v_{-i}}\left[x_i(v_i, v_{-i})\right]$

$$= \sum_i \mathbb{E}_{v_i\sim F_i}\left[v_i \cdot \hat{x}_i(v_i)\right] - \sum_i \int_{v_i=0}^{+\infty}\int_{t=0}^{v_i} \hat{x}_i(t)\cdot f_i(v_i)\,dt\,dv_i$$



$$= \sum_i \mathbb{E}_{v_i\sim F_i}\left[v_i \cdot \hat{x}_i(v_i)\right] - \sum_i \int_{t=0}^{+\infty}\int_{v_i=t}^{+\infty} \hat{x}_i(t)\, f_i(v_i)\,dt\,dv_i$$

$$= \sum_i \mathbb{E}_{v_i\sim F_i}\left[v_i \cdot \hat{x}_i(v_i)\right] - \sum_i \int_{t=0}^{+\infty} \hat{x}_i(t)\cdot(1-F_i(t))\,dt$$

$$= \sum_i \int_{v_i=0}^{+\infty} v_i \cdot \hat{x}_i(v_i)\, f(v_i)\,dv_i - \sum_i \int_{v_i=0}^{+\infty} \hat{x}_i(v_i)\,(1-F_i(v_i))\,dv_i$$

$$= \sum_i \int_{v_i=0}^{+\infty} \hat{x}_i(v_i) \cdot \left(v_i - \frac{1-F_i(v_i)}{f(v_i)} \right) f(v_i)\, dv_i$$

$$= \sum_i \mathbb{E}_{v_i}\left[\hat{x}_i(v_i) \cdot \varphi_i(v_i) \right] = \mathbb{E}_{v \sim F}\left[\sum_i x_i(v) \cdot \varphi_i(v_i) \right]$$

$\boxtimes$