Estimating the Loss of Efficiency due to Competition in Mobility on Demand Markets

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Overview

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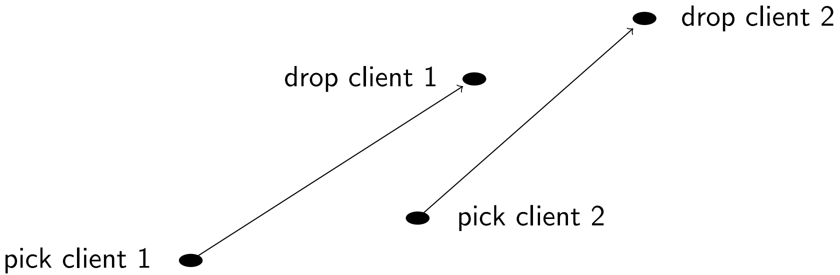
$\Rightarrow$ MoD systems has a great potential of being an efficient mean of transportation at a cheap price.

Issue of competition

Let's imagine the case of pooled demands :



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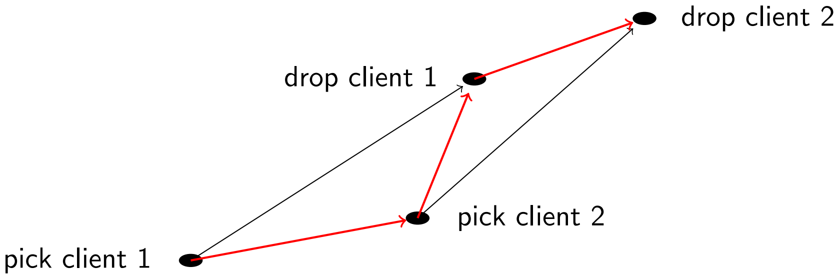

⇒ Competition might considerably undermine the efficiency of MoD systems.

Model

- The city is represented as a complete network of N stations.
- Hypothesis of an hourly steady state.
- Distance matrix $D = (d_{ij})_{(i,j)}$ and travel time $(\tau_{ij})_{(i,j)}$
- For each edge (i,j), customer demand $\Lambda = (\Lambda_{ij})$
- Node total demand throughput $\Lambda_i = \sum_j (\Lambda_{ji} - \Lambda_{ij}) = (A.\Lambda)_i$

We study the cost of rebalancing as a function $RC(\Lambda)$:

$$\begin{aligned} \min \quad & \sum_{(i,j)} \tau_{ij} x_{ij} \\ \text{s.t. } \forall i, \; & \sum_j (x_{ji} - x_{ij}) = \Lambda_i \\ & x \geq 0 \end{aligned}$$

First properties

Properties

- RC is equal to its dual :

$$\max \quad \sum_i \alpha_i . \Lambda_i$$
$$\text{s.t. } \forall (i,j),\ \alpha_i - \alpha_j \leq \tau_{ij}$$

- RC is convex.
- $\forall \beta$ corner point of the dual, $\forall c \in \mathbb{R}$, $\beta + c$ is also a corner point and it yields the same score as β.

Consequence: Without loss of generality, we can set any coordinate of β to zero.

Worst case study

Objective: Find the value of the worst split of demand in two parts, in order to check if the worst case is significant.

The Price of Fragmentation (PoF) is:

$$\begin{aligned} \max \quad & RC(\lambda) + RC(\Lambda - \lambda) \\ \text{s.t.} \quad & 0 \leq \lambda \leq \Lambda \end{aligned}$$

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Property

The PoF is convex. Thus, the optimal value is reached on a corner point.

Proof: Consequence of the convexity of RC.

Computing the worst PoF

- The PoF is a non-linear convex function, almost everywhere differentiable, over a bounded polyhedron isomorph to [0,1].

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- Finding the best corner point is NP-hard.
- Approximated evaluation via projected subgradient descent.

Result: Increase of 567% of the rebalancing cost due to the split.

PoF under Stochastic Demand Splitting

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$\Rightarrow$ Use a stochastic model to compute an average loss with respect to the monopoly, ie:

$$\gamma = \mathbb{E}[RC(\lambda) + RC(\Lambda - \lambda)] - RC(\Lambda)$$

With λ being a splitting r.v. respecting an exogenous market shares ratio ρ, first assumed to be homogeneous.

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Initial idea: Use a binomial split s.t. $\lambda \sim \mathrm{B}(\Lambda, \rho)$, assuming Λ has integer coordinates.

Properties of the Binomial process

Property: Fluid limit

Under binomial splitting, Let's define $\theta \in \mathbb{N}$ such that $\lambda^\theta \sim \mathsf{B}\left(\theta\Lambda, \rho\right)$. We have when $\theta \to \infty$:

$$\mathbb{E}[RC(\lambda^\theta)]/\theta = \mathbb{E}[RC(\lambda^\theta/\theta)] \to RC(\mathbb{E}[\lambda])$$

Thus, when the demand is scaled we get that $\gamma \to 0$.

Proof: Consequence of the central limit theorem, the continuity of g and the decomposition of binomial into a sum of bernoulli random variables.

Consequence of this property

The previous property implies that there is no loss of efficiency when the number of demands tends to infinity.

Questions

If the efficiency loss disappears when the demand is scaled to infinity, then how much does the loss depends on the demand volume? How fast do we converge to the fluid limit?

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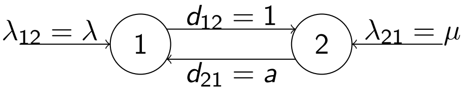
If the efficiency loss disappears when the demand is scaled to infinity, then how much does the loss depends on the demand volume? How fast do we converge to the fluid limit?

We will then focus on the rescaled PoF which depends on $\theta \in \mathbb{N}$:

$$\gamma^{\theta} = \frac{PoF(\theta)}{\theta} = \frac{\mathbb{E}[RC(\lambda^{\theta}) + RC(\theta\Lambda - \lambda^{\theta})] - RC(\theta\Lambda)}{\theta}$$

The example of the 2 nodes network

- Dual variables for each station $\alpha_1 = 0$ and α_2,
- Total demand $\lambda_{12} = \lambda$ and $\lambda_{21} = \mu$,
- Random split $X \sim \mathsf{N}(\theta\rho\lambda, \theta\lambda s)$ and $Y \sim \mathsf{N}(\theta\rho\mu, \theta\mu s)$, $s = \rho(1-\rho)$
- Distances by $d_{12} = 1$ and $d_{21} = a \geq 1$.



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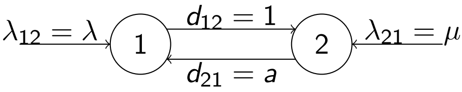
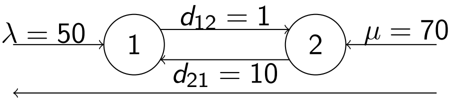


Formula: *RC* for one company

$$RC(\lambda,\mu) = \max\{a.(\mu-\lambda), (\lambda-\mu)\}$$
$$= [\,(a-1).(\mu-\lambda) + (a+1).|\mu-\lambda|\,]/2$$

Two numerical examples

- Let's set $\lambda = 50$, $\mu = 70$, $a = 10$, $\rho = 0.5$
- Then $\mathbb{E}[X] = 25$, $\mathbb{E}[Y] = 35$
- $\gamma = g(X, Y) + g(\lambda - X, \mu - Y) - g(\lambda, \mu)$



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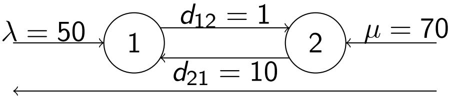

Example 1: $X = 30$ and $Y = 40$

- Company 1: $g(30, 40) = 10 * 10 = 100$
- Company 2: $g(20, 30) = 10 * 10 = 100$
- Monopoly: $g(50, 70) = 200$

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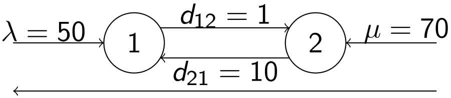

Example 2: $X = 30$ and $Y = 25$

- Company 1: $g(30, 25) = 1 * 5 = 5$
- Company 2: $g(20, 45) = 10 * 25 = 250$
- Monopoly: $g(50, 70) = 200$

2 Nodes Network: Theoretical result

Theorem

When $\theta \to \infty$, $\exists A \in \mathbb{R}$ such that :

- $\lambda = \mu \;\Rightarrow\; \gamma^\theta = A\theta^{-1/2}$
- $\lambda \neq \mu \;\Rightarrow \gamma^\theta = A\theta^{-3/2} e^{-\frac{\rho(\lambda-\mu)^2}{2(1-\rho)(\lambda+\mu)}.\theta} + o(\theta^{-3/2} e^{-\frac{\rho(\lambda-\mu)^2}{2(1-\rho)(\lambda+\mu)}.\theta})$

Proof: Calculus via the folded normal distribution.

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- We distinguish two regimes:
  - Balanced demands: square root decay.
  - Imbalanced demands: Exponential decay.

PoF in General Networks

Since the change of corner point induces the loss of efficiency we thus define:

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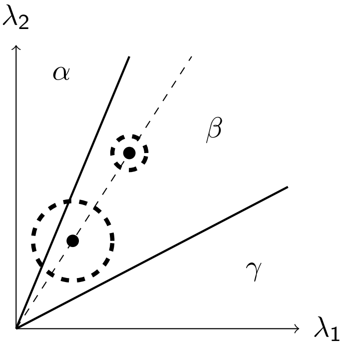
Property

$\forall \; \alpha \in \mathcal{E}, \; C_\alpha$ is such that :

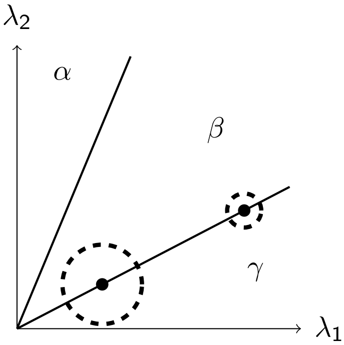
- It is a closed cone.
- The intersection of two cones is a plane.

Proof: The cones structure comes from the homogeneity of RC. The closedness is due to the fact that $C_\alpha = (RC(\lambda) - \alpha^\intercal \lambda)^{-1}(\{0\})$.

Geometrical intuition



Exponential decay



Square root decay

Main Theorem: Concepts

- ρ is not necessarily homogeneous.
- $\rho \odot \Lambda \in C_\alpha$, $(1-\rho) \odot \Lambda \in C_\beta$ and $\Lambda \in C_\eta$, i.e. α, β and η are the optimal corner points for respectively company 1, company 2 and the monopoly.
- The function f used in the main theorem will denote a speed decay which is faster than a square root (i.e. the exponential decay).

Main Theorem

Theorem

Let's suppose that:

- There is a r.v. ξ such that $\lambda^\theta = \theta\rho \odot \Lambda + \sqrt{\theta}\sigma\xi$ with $\mathbb{E}[||\xi||_1] < \infty$
- $\mathbb{P}(|\xi| > t) = \mathcal{O}(f(t))$ with $f(t) = \mathcal{O}(t)$

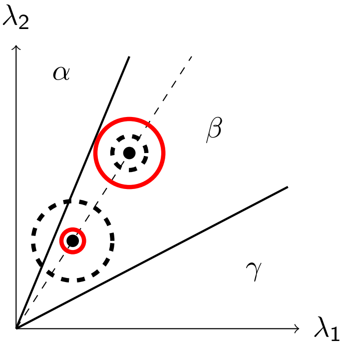
Let's define:

$$L = \alpha^\intercal \rho \odot \Lambda + \beta^\intercal (1-\rho) \odot \Lambda - \eta^\intercal \Lambda$$

Then we have:

- $\alpha \in \mathring{C}_\alpha$ *and* $\beta \in \mathring{C}_\beta \Rightarrow \gamma^\theta = L + \mathcal{O}(f(\sqrt{\theta}))$
- $\alpha \in C_\alpha \setminus \mathring{C}_\alpha$ *or* $\beta \in C_\beta \setminus \mathring{C}_\beta \Rightarrow \gamma^\theta = L + \Theta(\theta^{-1/2})$

Intuition of the proof



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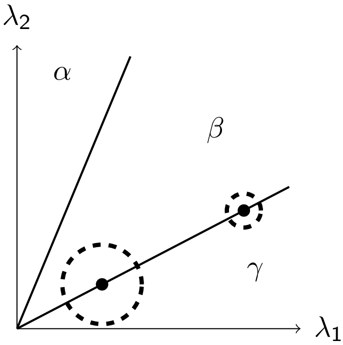

Square root decay

Interesting cases

- Applies for both the binomial process and the Poisson process:
  - Binomial process: $\exists a \in \mathbb{R},\ \mathbb{P}(|\xi| > t) = \mathcal{O}(e^{-at^2})$
  - Poisson process: $\exists a \in \mathbb{R},\ \mathbb{P}(|\xi| > t) = \mathcal{O}(e^{-at})$

- If ρ is homogenous, then $\alpha = \beta = \eta \ \Rightarrow\ L = 0$

Sufficient condition on the square root regime

We know that the square root regime is critical because it decays much slower. **But When does it happen?**

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Theorem: Sufficient condition for the square root decay

Let's suppose that:

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- $E_H = \{(i,j) |\ x_{ij} > 0\}$ contains at least two connected components.

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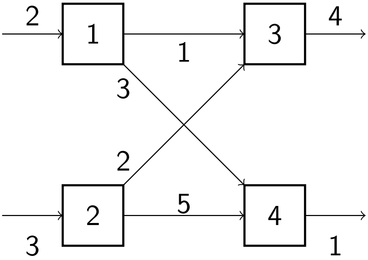
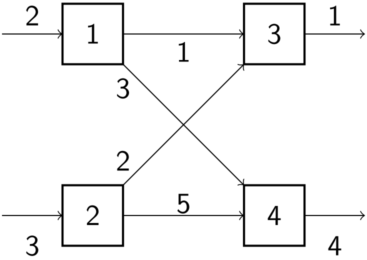
Then the dual optimal point is defined by two different corner points.

Proof: Let's define $E_{\hat{H}} = \{(i,j)|\ \alpha_i^* - \alpha_j^* = \tau_{ij}\}$. Starting from $E_{\hat{H}} = E_H$, we define a dual corner point β which can be slightly modified to saturate new dual constraints and defining another dual corner point.

Example: an imbalanced graph with square root decay



$$\Lambda^1 = [2, 3, -4, -1]$$



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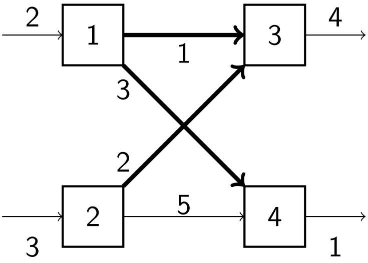

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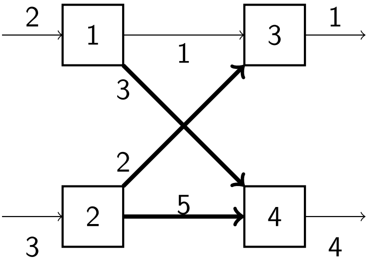

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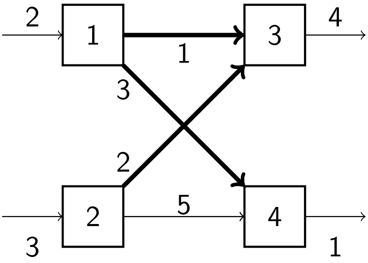

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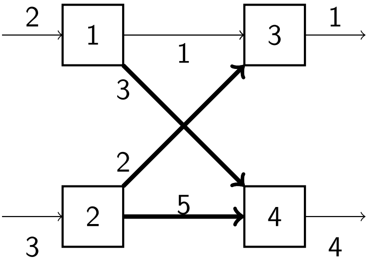
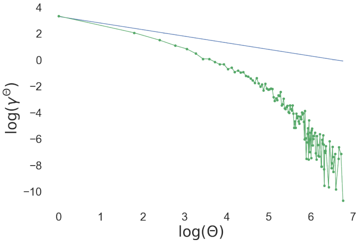


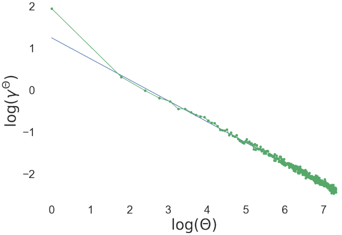
$\Lambda^2 = [2, 3, -1, -4]$

$\Rightarrow \Lambda^3 = [2, 3, -3, -2]$ is optimal for two corner points.

Numerical simulations with the previous network

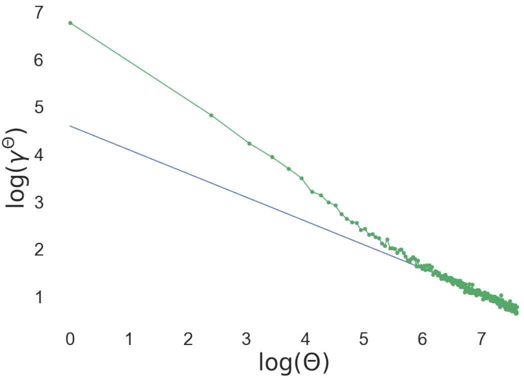


Simulations with Λ^1:
Imbalanced demand.



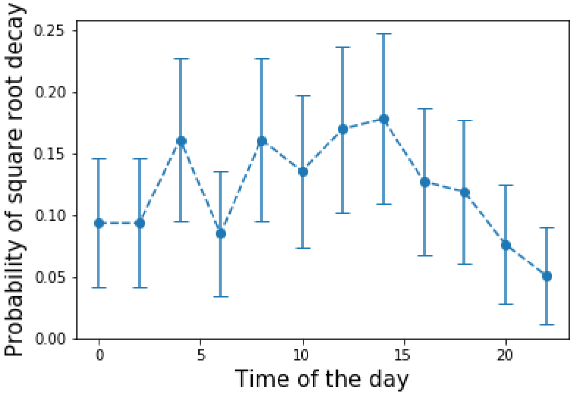
Simulations with Λ^3:
Balanced demand.

Numerical simulations with real data



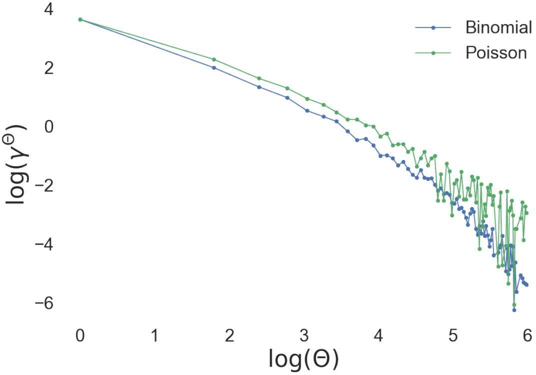
$\log(\gamma^{\theta})$ depending on $\log(\theta)$. TLC Data clustered into 40 stations.

Frequency of a square root decay



Probability of having two connected components on two months period from TLC dataset.

Numerical simulations comparing random processes



$\log(\gamma^{\theta})$ depending on $\log(\theta)$. Comparison of binomial and Poisson processes.

Those properties allow to draw conclusions on how we can best treat the total demand. We have those three different regimes:

On the necessity of a market maker

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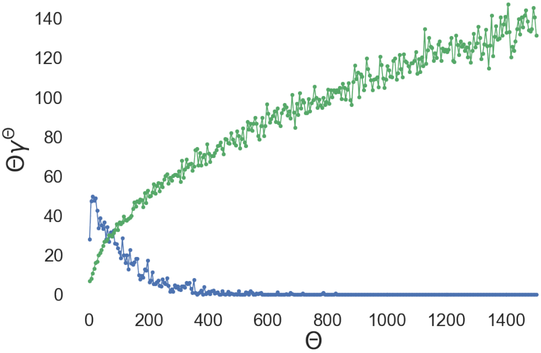
Thus we deduce that to enhance the efficiency we need to:

- Homogenize demands between companies.
- Imbalance demands at each node for both companies.

Numerical simulations of the two decaying regimes



$\theta\gamma^\theta$ depending on θ for the demands Λ^1 and Λ^3

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But is it possible to transform a balanced demand into two imbalanced one?

- We have that $\Lambda_1 + \Lambda_2 = \Lambda$, and they need to keep the same corner point to have at most a square root decay.
- Assuming the cones are convex, if they have the same corner points then so has the monopoly.
- Conversely, if the monopoly is optimized on several corner points, thus each company should be optimized on different corner points.

On the necessity of a market maker

Though, pricing policies might already aim at balancing demands, thus generating balanced nodes and inducing a square root decay.
But is it possible to transform a balanced demand into two imbalanced one?

- We have that $\Lambda_1 + \Lambda_2 = \Lambda$, and they need to keep the same corner point to have at most a square root decay.
- Assuming the cones are convex, if they have the same corner points then so has the monopoly.
- Conversely, if the monopoly is optimized on several corner points, thus each company should be optimized on different corner points.

Conclusion: It seems that the demand itself needs to be modified so as to make it imbalanced.

Ideas to go further

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- Queuing theory to study the availability of nodes.
- On-line Linear Programs to have an unsteady demand.
- Lumped model instead of clustering the city into stations.
- Game theoretical models to consider pricing or social welfare.

Conclusion

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Thank you !

Proof for the exponential decay (1/3)

We suppose for sake of simplicity that ρ is homogenous.
We define $x = \rho\Lambda$, $y = (1-\rho)\Lambda$, and δ such that:

$$\delta = \min\{\min_{\alpha\in\mathcal{E}} d(x, \mathcal{P}_{(\alpha,\beta)}),\ \min_{\alpha\in\mathcal{E}} d(y, \mathcal{P}_{(\alpha,\beta)})\}$$

Where $\mathcal{P}_{(\alpha,\beta)} = C_\alpha \cap C_\beta$.

We also denote:

$$\bar{\eta} = \max_{\eta\in\mathcal{E}} \|\eta\|_\infty$$

Furthermore we have: $\lambda^\theta \in C_{\alpha_1}$ and $(\theta\Lambda - \lambda^\theta) \in C_{\alpha_2}$

Proof for the exponential decay (2/3)

$$\gamma^\theta = \theta^{-1}.\mathbb{E}[\alpha_1^\intercal \lambda^\theta + \alpha_2^\intercal(\theta\Lambda - \lambda^\theta) - \eta^\intercal \theta \Lambda]$$

$$= \theta^{-1}.\mathbb{E}[(\rho\alpha_1 + (1-\rho)\alpha_2 - \eta)^\intercal \theta\Lambda + (\alpha_1 - \alpha_2)^\intercal \sqrt{\theta}\sigma\xi]$$

$$\leq 2\bar{\eta}.(\sum_i |\Lambda_i|).\mathbb{P}(\exists i,\ |(\sqrt{\theta}\sigma\xi)_i| > \theta\delta)$$
$$+ \theta^{-1}||\alpha_1 - \alpha_2||_\infty.\mathbb{E}[\ ||\sqrt{\theta}\sigma\xi||_1.\mathbb{1}_{\{\exists i,\ |(\sqrt{\theta}\sigma\xi)_i| > \theta\delta\}}]$$

$$\leq 2\bar{\eta}.(\sum_i |\Lambda_i|).\mathbb{P}(\exists i,\ |(\sqrt{\theta}\sigma\xi)_i| > \theta\delta)$$
$$+ 2\theta^{-1}\bar{\eta}||\sqrt{\theta}\sigma||_1.\mathbb{E}[\ ||\xi||_1 \mid \exists i,\ |(\sqrt{\theta}\sigma\xi)_i| > \theta\delta]$$
$$.\mathbb{P}(\exists i,\ |(\sqrt{\theta}\sigma\xi)_i| > \theta\delta)$$

Proof for the exponential decay (3/3)

Thanks to the law of total expectation:

$$\mathbb{E}[\ ||\xi||_1 \mid \exists i,\ |(\sqrt{\theta}\sigma\xi)_i| > \theta\delta] < \infty$$

Furthermore:

$$\begin{aligned}\mathbb{P}(\exists i,\ |(\sqrt{\theta}\sigma\xi)_i| > \theta\delta) &\leq \mathbb{P}(\exists i,\ \sqrt{\theta}||\sigma||_1.|\xi_i| > \theta\delta) \\ &\leq N^2.\mathbb{P}(||\sigma||_1.|\xi_i| > \delta\sqrt{\theta}) \\ &= \mathcal{O}(f(\sqrt{\theta}))\end{aligned}$$

Proof for the square root decay (1/2)

Demonstration of the lower bound:

$$\gamma^\theta = \theta^{-1}.\mathbb{E}[g(\lambda^\theta) + g(\theta\Lambda - \lambda^\theta) - g(\theta\Lambda)]$$

$$\geq \theta^{-1}.\mathbb{E}[(\alpha^\intercal\lambda^\theta + \beta^\intercal(\theta\Lambda - \lambda^\theta) - \beta^\intercal\theta\Lambda).\mathbb{1}_{\{\alpha^\intercal\sigma\xi \geq \beta^\intercal\sigma\xi\}}]$$

$$+ \theta^{-1}.\mathbb{E}[(\beta^\intercal\lambda^\theta + \alpha^\intercal(\theta\Lambda - \lambda^\theta) - \alpha^\intercal\theta\Lambda).\mathbb{1}_{\{\beta^\intercal\sigma\xi \geq \alpha^\intercal\sigma\xi\}}]$$

$$= \theta^{-1/2}.\mathbb{E}[(\alpha - \beta)^\intercal\sigma\xi.\mathbb{1}_{\{\alpha^\intercal\sigma\xi \geq \beta^\intercal\sigma\xi\}}]$$

$$+ \theta^{-1/2}.\mathbb{E}[(\alpha - \beta)^\intercal\sigma\xi.\mathbb{1}_{\{\beta^\intercal\sigma\xi \geq \alpha^\intercal\sigma\xi\}}]$$

$$= \theta^{-1/2}.\mathbb{E}[|(\alpha - \beta)^\intercal\sigma\xi|]$$

$$= \Omega(\theta^{-1/2})$$

Proof for the square root decay (2/2)

Demonstration of the higher bound:

$$\begin{aligned}
\gamma^\theta &= \theta^{-1}.\mathbb{E}[(g(\lambda^\theta) + g(\theta\Lambda - \lambda^\theta) - g(\theta\Lambda))\mathbb{1}_{\{\forall i, (\sqrt{\theta}\sigma\xi)_i \leq \theta\delta\}}] \\
&\quad + \theta^{-1}.\mathbb{E}[(g(\lambda^\theta) + g(\theta\Lambda - \lambda^\theta) - g(\theta\Lambda))\mathbb{1}_{\{\exists i, (\sqrt{\theta}\sigma\xi)_i > \theta\delta\}}] \\
&\leq \theta^{-1/2}.\mathbb{E}[|(\alpha - \beta)^\intercal \sigma\xi|\mathbb{1}_{\{\forall i, (\sqrt{\theta}\sigma\xi)_i \leq \theta\delta\}}] + \mathcal{O}(f(\sqrt{\theta})) \\
&\leq \theta^{-1/2}.\mathbb{E}[|(\alpha - \beta)^\intercal \sigma\xi|] + \mathcal{O}(f(\sqrt{\theta})) \\
&= \mathcal{O}(\theta^{-1/2})
\end{aligned}$$

Thus we finally have: $\gamma^\theta = \Theta(\sqrt{\theta})$