Semidefinite programming lifts and sparse sums of squares

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Joint work with João Gouveia (Coimbra), Pablo Parrilo (MIT),
Richard Robinson (Microsoft), James Saunderson (Monash), Rekha Thomas (UW)

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Representation of convex sets

- Central question in optimization is to optimize a linear function ℓ on a convex set C:

$$\min_{x \in C} \ell(x).$$

- Need "good" description of C to solve optimization problem efficiently.
- Conic programming descriptions

Semidefinite representation

- Feasible set of a semidefinite program:

$$\begin{cases} X \succeq 0 \text{ (positive semidefinite constraint)} \\ \mathcal{A}(X) = b \text{ (linear constraints)} \end{cases}$$

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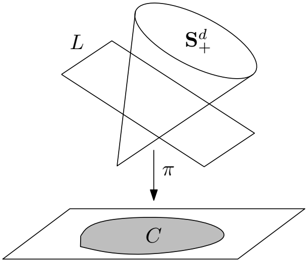
- Convex set C has a **semidefinite representation of size** d if:

$$C = \pi(\mathbf{S}_+^d \cap L)$$

$\mathbf{S}_+^d = d \times d$ positive semidefinite matrices

$L =$ affine subspace

$\pi =$ linear map

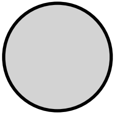


Examples of semidefinite representations

Examples:

- Disk in $\mathbb{R}^2$ has a SDP representation of size 2

$$x^2 + y^2 \leq 1 \quad \Leftrightarrow \quad \begin{bmatrix} 1 - x & y \\ y & 1 + x \end{bmatrix} \succeq 0$$



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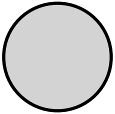
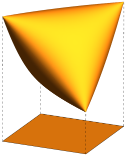


- Square $[-1, 1]^2$ has a SDP representation of size 3

$$[-1, 1]^2 = \left\{ (x_1, x_2) \in \mathbb{R}^2 : \exists u \in \mathbb{R} \begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & u \\ x_2 & u & 1 \end{bmatrix} \succeq 0 \right\}$$



Existential question vs. complexity question

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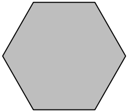
- Complexity question: Given a convex set C, what is **smallest** semidefinite representation of C? $\rightarrow$ **Positive semidefinite rank**

Importance of lifting



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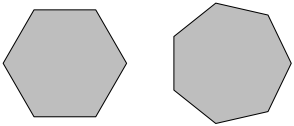

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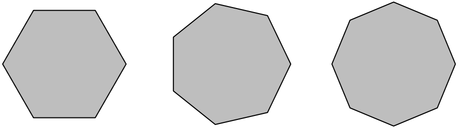

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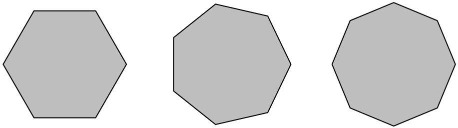

Ben-Tal and Nemirovski: Regular polygon with 2^n sides can be described using only $\approx n$ inequalities!

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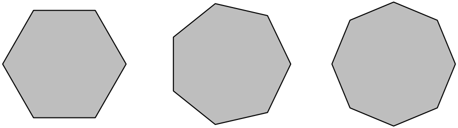

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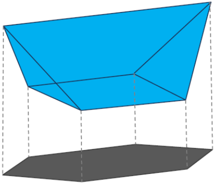


Lift = “inverse” of elimination (Fourier-Motzkin, Tarski-Seidenberg, ...)

Lifts of polytopes and ranks of matrices

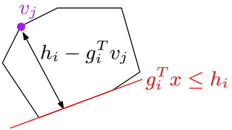
P polytope in $\mathbb{R}^d$

Slack matrix of P: matrix of size $\#\text{facets}(P) \times \#\text{vertices}(P)$:

$$M_{i,j} = h_i - g_i^T v_j$$

where

- $g_i^T x \leq h_i$ are the facet inequalities of P
- v_j are the vertices of P



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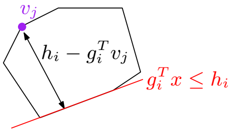

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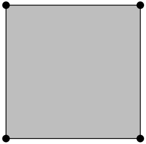

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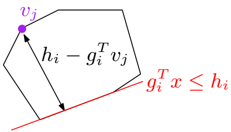
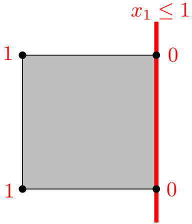


0 1 1 0



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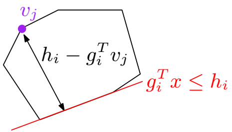

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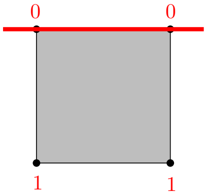

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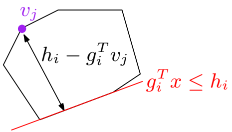

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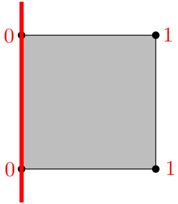

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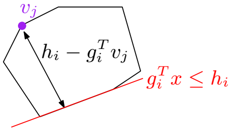

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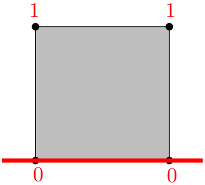


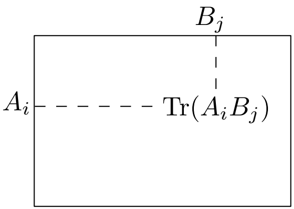
Positive semidefinite rank

$M \in \mathbb{R}^{p \times q}$ with nonnegative entries

- Positive semidefinite factorization:

$$M_{ij} = \mathrm{Tr}(A_i B_j), \quad \text{where} \quad A_i, B_j \in \mathbf{S}_+^k$$

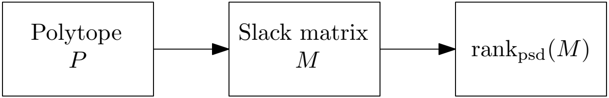
- $\mathrm{rank}_{\mathrm{psd}}(M)$ = size of smallest psd factorization



SDP representations and psd rank

Theorem (Gouveia, Parrilo, Thomas, 2011)

Let P be polytope with slack matrix M. The smallest semidefinite representation of P has size exactly $\mathrm{rank}_{\mathrm{psd}}(M)$.



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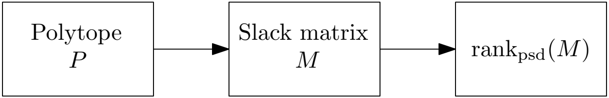

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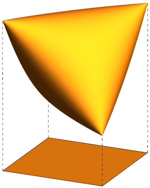
Example:

- Slack matrix of square $[-1,1]^2$ has positive semidefinite rank 3.
- Can show that any LP lift of square has size 4



LP lifts vs. SDP lifts

Question: How powerful are SDP lifts compared to LP lifts?

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Theorem (Fawzi, Saunderson, Parrilo, 2015)

There is a family of polytopes $P_d \subset \mathbb{R}^{2d}$ *such that*

$$\frac{\text{rank}_{\text{psd}}(P_d)}{\text{rank}_{LP}(P_d)} \leq O\left(\frac{\log d}{d}\right) \to 0.$$

- P_d = trigonometric cyclic polytope (generalization of regular polygons)
- Construction uses tools from Fourier analysis + *sparse sums of squares*

Trigonometric cyclic polytopes

Regular N-gon (N roots of unity)

$$TC_{N,1} = \text{conv}\left\{ e^{2i\pi x/N} : x \in \mathbb{Z}_N \right\} \subset \mathbb{C} \cong \mathbb{R}^2$$

Trigonometric cyclic polytopes

$$TC_{N,2} = \mathrm{conv}\left\{\left(e^{2i\pi x/N}, e^{2i\pi(2x)/N}\right) : x \in \mathbb{Z}_N\right\} \subset \mathbb{C}^2 \cong \mathbb{R}^4$$

Trigonometric cyclic polytopes

$$TC_{N,3} = \text{conv}\left\{\left(e^{2i\pi x/N}, e^{2i\pi(2x)/N}, e^{2i\pi(3x)/N}\right) : x \in \mathbb{Z}_N\right\} \subset \mathbb{C}^3 \cong \mathbb{R}^6$$

Trigonometric cyclic polytopes

Degree d trigonometric cyclic polytope

$$TC_{N,d} = \operatorname{conv}\left\{ \left(e^{2i\pi x/N}, e^{4i\pi x/N}, \ldots, e^{2i\pi dx/N}\right) : x \in \mathbb{Z}_N \right\} \subset \mathbb{C}^d \cong \mathbb{R}^{2d}$$

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- $2d$-dimensional generalization of regular N-gons

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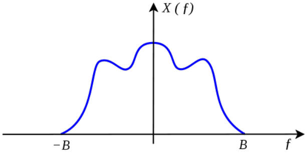
- $2d$-dimensional generalization of regular N-gons
- Slack matrix has nice structure
- Rows are bandlimited with bandwidth d (i.e., their Fourier transform is supported on $\{-d, \ldots, d\}$)



SDP lifts and sums-of-squares

M = slack matrix of trigonometric cyclic polytope $TC_{N,d}$

Proposition

Assume there is a subspace $V \subset \mathbb{R}^N$ such that each row ℓ of M can be written as a sum of squares of elements in V, i.e.,

$$\ell = \sum_{j=1}^{J} f_j^2 \quad \textit{where} \quad f_j \in V \ \ (j = 1, \dots, J)$$

Then M has a SDP factorization of size $\dim V$.

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Key point: constructing SDP factorization boils down to finding subspace V

Note: Lasserre hierarchy corresponds to $V =$ polynomials of degree $\leq k$

Sparse sums-of-squares

How to look for subspace V of $\mathbb{R}^N$? Use Fourier analysis:

- Signals of length N decompose into Fourier basis:

$$\mathbb{C}^N = \bigoplus_{k \in \mathbb{Z}_N} \mathbb{C} e_k$$

  where

$$e_k(x) = e^{2ik\pi x/N}.$$

- Main idea: Find subspace of the form

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- Main idea: Find subspace of the form

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  where K is small. V = vectors with sparse Fourier transform
- Fourier-analytic question: Find a small K such that any *nonnegative bandlimited vector with bandwidth d* has a sum-of-squares representation with functions $f_j \in V$

Result

Using graph-theoretic tools we show:

Theorem

If d divides N then $\text{rank}_{\text{psd}}(TC_{N,d}) \leq 3d\log(N/d)$.

$\Rightarrow$ SDP factorization of size $\leq 3d\log(N/d)$ of the slack matrix of $TC_{N,d}$.

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Gap is obtained in regime $N = d^2$:

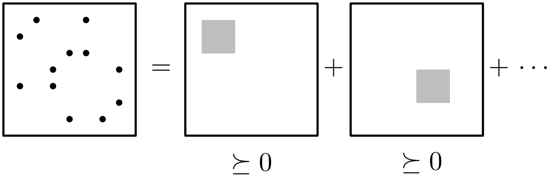
- SDP lift has size $3d\log(d)$
- $\mathrm{rank}_{LP}(TC_{d^2,d}) \geq d^2$ [Fiorini et al.]

One ingredient of proof: sparse psd matrices

Positive semidefinite matrices with chordal sparsity

Theorem (Grone, Johnson, Sá, Wolkowicz, 1984; Griewank, Toint 1984)

If $Q \succeq 0$ and sparse w.r.t. chordal graph Γ, then Q decomposes as sum of psd matrices each supported on a maximal clique of Γ.



Conclusion

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- Connection with matrix factorization and sums of squares
- Linear programming vs. semidefinite programming lifts for polytopes

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