Semidefinite programming lifts and sparse sums of squares

Hamza Fawzi (University of Cambridge)

Joint work with João Gouveia (Coimbra), Pablo Parrilo (MIT), Richard Robinson (Microsoft), James Saunderson (Monash), Rekha Thomas (UW)

PGMO Days 2016
Central question in optimization is to optimize a linear function $\ell$ on a convex set $C$:

$$\min_{x \in C} \ell(x).$$

Need “good” description of $C$ to solve optimization problem efficiently.

Conic programming descriptions
Semidefinite representation

- Feasible set of a semidefinite program:

\[
\begin{align*}
X &\succeq 0 \text{ (positive semidefinite constraint)} \\
\mathcal{A}(X) &= b \text{ (linear constraints)}
\end{align*}
\]
Semidefinite representation

- Feasible set of a semidefinite program:

\[
\begin{cases}
X \succeq 0 \text{ (positive semidefinite constraint)} \\
A(X) = b \text{ (linear constraints)}
\end{cases}
\]

- Convex set $C$ has a **semidefinite representation of size $d$** if:

\[
C = \pi(S^d_+ \cap L)
\]

$S^d_+ = d \times d$ positive semidefinite matrices

$L =$ affine subspace

$\pi = $ linear map
Examples of semidefinite representations

Examples:
- Disk in $\mathbb{R}^2$ has a SDP representation of size 2

$$x^2 + y^2 \leq 1 \iff \begin{bmatrix} 1 - x & y \\ y & 1 + x \end{bmatrix} \succeq 0$$
Examples of semidefinite representations

Examples:

- Disk in $\mathbb{R}^2$ has a SDP representation of size 2

\[ x^2 + y^2 \leq 1 \iff \begin{bmatrix} 1 - x & y \\ y & 1 + x \end{bmatrix} \succeq 0 \]

- Square $[-1, 1]^2$ has a SDP representation of size 3

\[ [-1, 1]^2 = \left\{ (x_1, x_2) \in \mathbb{R}^2 : \exists u \in \mathbb{R} \begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & u \\ x_2 & u & 1 \end{bmatrix} \succeq 0 \right\} \]
Existential question vs. complexity question

- **Existential question**: Which convex sets admit a semidefinite representation?

  *Helton-Nie conjecture*: Any convex set defined using polynomial inequalities has a semidefinite representation
Existential question vs. complexity question

- **Existential question**: Which convex sets admit a semidefinite representation?

  *Helton-Nie conjecture*: Any convex set defined using polynomial inequalities has a semidefinite representation

- **Complexity question**: Given a convex set $C$, what is smallest semidefinite representation of $C$?
Existential question vs. complexity question

- **Existential question**: Which convex sets admit a semidefinite representation?

  *Helton-Nie conjecture*: Any convex set defined using polynomial inequalities has a semidefinite representation

- **Complexity question**: Given a convex set $C$, what is\textit{ smallest} semidefinite representation of $C$? \rightarrow \textbf{Positive semidefinite rank}
Importance of lifting

Regular polygon with $2^n$ sides can be described using only $\approx n$ inequalities!

Lift = “inverse” of elimination (Fourier-Motzkin, Tarski-Seidenberg, ...)

6/16
Importance of lifting

Ben-Tal and Nemirovski: Regular polygon with \(2^n\) sides can be described using \(\approx n\) inequalities!

Lift = "inverse" of elimination (Fourier-Motzkin, Tarski-Seidenberg, ...)

6/16
Importance of lifting

Regular polygon with $2^n$ sides can be described using only $\approx n$ inequalities!

Lift = "inverse" of elimination (Fourier-Motzkin, Tarski-Seidenberg, ...)
Importance of lifting

*Ben-Tal and Nemirovski*: Regular polygon with $2^n$ sides can be described using only $\approx n$ inequalities!
Importance of lifting

*Ben-Tal and Nemirovski:* Regular polygon with $2^n$ sides can be described using only $\approx n$ inequalities!

Lift = “inverse” of elimination (Fourier-Motzkin, Tarski-Seidenberg, ...
Lifts of polytopes and ranks of matrices

$P$ polytope in $\mathbb{R}^d$

Slack matrix of $P$: matrix of size $\#\text{facets}(P) \times \#\text{vertices}(P)$:

$$M_{i,j} = h_i - g_i^T v_j$$

where

- $g_i^T \mathbf{x} \leq h_i$ are the facet inequalities of $P$
- $v_j$ are the vertices of $P$
Lifts of polytopes and ranks of matrices

\( P \) polytope in \( \mathbb{R}^d \)

Slack matrix of \( P \): matrix of size \#facets(\( P \)) \( \times \) \#vertices(\( P \)):

\[
M_{i,j} = h_i - g_i^T v_j
\]

where

- \( g_i^T x \leq h_i \) are the facet inequalities of \( P \)
- \( v_j \) are the vertices of \( P \)
Lifts of polytopes and ranks of matrices

$P$ polytope in $\mathbb{R}^d$

Slack matrix of $P$: matrix of size $\#\text{facets}(P) \times \#\text{vertices}(P)$:

$$M_{i,j} = h_i - g_i^T v_j$$

where

- $g_i^T x \leq h_i$ are the facet inequalities of $P$
- $v_j$ are the vertices of $P$
Lifts of polytopes and ranks of matrices

$P$ polytope in $\mathbb{R}^d$

Slack matrix of $P$: matrix of size $\#\text{facets}(P) \times \#\text{vertices}(P)$:

$$M_{i,j} = h_i - g_i^T v_j$$

where

- $g_i^T x \leq h_i$ are the facet inequalities of $P$
- $v_j$ are the vertices of $P$
Lifts of polytopes and ranks of matrices

$P$ polytope in $\mathbb{R}^d$

Slack matrix of $P$: matrix of size $\#\text{facets}(P) \times \#\text{vertices}(P)$:

\[
M_{i,j} = h_i - g_i^T v_j
\]

where

- $g_i^T x \leq h_i$ are the facet inequalities of $P$
- $v_j$ are the vertices of $P$
Lifts of polytopes and ranks of matrices

$P$ polytope in $\mathbb{R}^d$

Slack matrix of $P$: matrix of size $\#\text{facets}(P) \times \#\text{vertices}(P)$:

$$M_{i,j} = h_i - g_i^T v_j$$

where

- $g_i^T x \leq h_i$ are the facet inequalities of $P$
- $v_j$ are the vertices of $P$
Positive semidefinite rank

\( M \in \mathbb{R}^{p \times q} \) with nonnegative entries

- **Positive semidefinite factorization:**

  \[ M_{ij} = \text{Tr}(A_i B_j), \quad \text{where} \quad A_i, B_j \in S_+^k \]

- \( \text{rank}_{\text{psd}}(M) = \) size of smallest psd factorization

\[ B_j \]

\[ A_i \]

\[ \text{Tr}(A_i B_j) \]
Theorem (Gouveia, Parrilo, Thomas, 2011)

Let \( P \) be polytope with slack matrix \( M \). The smallest semidefinite representation of \( P \) has size exactly \( \text{rank}_{\text{psd}}(M) \).

- Works more generally for convex sets (slack matrix is infinite)
- Proof based on duality for semidefinite programming
Theorem (Gouveia, Parrilo, Thomas, 2011)

Let $P$ be polytope with slack matrix $M$. The smallest semidefinite representation of $P$ has size exactly $\text{rank}_{\text{psd}}(M)$.

- Works more generally for convex sets (slack matrix is infinite)
- Proof based on duality for semidefinite programming

Example:
- Slack matrix of square $[-1, 1]^2$ has positive semidefinite rank 3.
- Can show that any LP lift of square has size 4
Question: How powerful are SDP lifts compared to LP lifts?

Theorem (Fawzi, Saunderson, Parrilo, 2015)
There is a family of polytopes $P_d \subset \mathbb{R}^d$ such that $\text{rank}_{\text{psd}}(P_d) \leq O(\log d) \to 0$.

$P_d$ = trigonometric cyclic polytope (generalization of regular polygons)

Construction uses tools from Fourier analysis + sparse sums of squares
**Question:** How powerful are SDP lifts compared to LP lifts?

**Theorem (Fawzi, Saunderson, Parrilo, 2015)**

There is a family of polytopes $P_d \subseteq \mathbb{R}^{2d}$ such that

\[
\frac{\text{rank}_{\text{psd}}(P_d)}{\text{rank}_{\text{LP}}(P_d)} \leq O\left(\frac{\log d}{d}\right) \to 0.
\]

- $P_d =$ trigonometric cyclic polytope (generalization of regular polygons)
- Construction uses tools from Fourier analysis + *sparse sums of squares*
Trigonometric cyclic polytopes

Regular $N$-gon ($N$ roots of unity)

$$TC_{N,1} = \text{conv}\left\{e^{2i\pi x/N} : x \in \mathbb{Z}_N\right\} \subset \mathbb{C} \cong \mathbb{R}^2$$
Trigonometric cyclic polytopes

\[ TC_{N,2} = \text{conv}\left\{ \left( e^{2i\pi x/N}, e^{2i\pi (2x)/N} \right) : x \in \mathbb{Z}_N \right\} \subset \mathbb{C}^2 \cong \mathbb{R}^4 \]
Trigonometric cyclic polytopes

\[ TC_{N,3} = \text{conv}\left\{ \left( e^{2i\pi x/N}, e^{2i\pi (2x)/N}, e^{2i\pi (3x)/N} \right) : x \in \mathbb{Z}_N \right\} \subset \mathbb{C}^3 \cong \mathbb{R}^6 \]
Trigonometric cyclic polytopes

Degree $d$ trigonometric cyclic polytope

$$TC_{N,d} = \text{conv}\left\{ \left( e^{2i\pi x/N}, e^{4i\pi x/N}, \ldots, e^{2i\pi d x/N} \right) : x \in \mathbb{Z}_N \right\} \subset \mathbb{C}^d \cong \mathbb{R}^{2d}$$
Trigonometric cyclic polytopes

Degree $d$ trigonometric cyclic polytope

$$TC_{N,d} = \text{conv}\left\{\left(e^{2i\pi x/N}, e^{4i\pi x/N}, \ldots, e^{2i\pi dx/N}\right) : x \in \mathbb{Z}_N\right\} \subset \mathbb{C}^d \cong \mathbb{R}^{2d}$$

- 2$d$-dimensional generalization of regular $N$-gons
Trigonometric cyclic polytopes

Degree $d$ trigonometric cyclic polytope

$$TC_{N,d} = \text{conv}\left\{\left(e^{2i\pi x/N}, e^{4i\pi x/N}, \ldots, e^{2i\pi dx/N}\right) : x \in \mathbb{Z}_N\right\} \subset \mathbb{C}^d \cong \mathbb{R}^{2d}$$

- $2d$-dimensional generalization of regular $N$-gons
- Slack matrix has nice structure
Trigonometric cyclic polytopes

Degree $d$ trigonometric cyclic polytope

$$TC_{N,d} = \text{conv}\left\{ \left( e^{2i\pi x/N}, e^{4i\pi x/N}, \ldots, e^{2i\pi dx/N} \right) : x \in \mathbb{Z}_N \right\} \subset \mathbb{C}^d \cong \mathbb{R}^{2d}$$

- $2d$-dimensional generalization of regular $N$-gons
- Slack matrix has nice structure
- Rows are bandlimited with bandwidth $d$ (i.e., their Fourier transform is supported on $\{-d, \ldots, d\}$)
SDP lifts and sums-of-squares

\[ M = \text{slack matrix of trigonometric cyclic polytope } TC_{N,d} \]

**Proposition**

Assume there is a subspace \( V \subset \mathbb{R}^N \) such that each row \( \ell \) of \( M \) can be written as a sum of squares of elements in \( V \), i.e.,

\[ \ell = \sum_{j=1}^{J} f_j^2 \quad \text{where} \quad f_j \in V \ (j = 1, \ldots, J) \]

Then \( M \) has a SDP factorization of size \( \dim V \).
$M$ = slack matrix of trigonometric cyclic polytope $TC_{N,d}$

**Proposition**

Assume there is a subspace $V \subset \mathbb{R}^N$ such that each row $\ell$ of $M$ can be written as a sum of squares of elements in $V$, i.e.,

$$\ell = \sum_{j=1}^{J} f_j^2 \quad \text{where} \quad f_j \in V \ (j = 1, \ldots, J)$$

Then $M$ has a SDP factorization of size $\dim V$.

**Key point:** constructing SDP factorization boils down to finding subspace $V$

**Note:** Lasserre hierarchy corresponds to $V = \text{polynomials of degree} \leq k$
Sparse sums-of-squares

How to look for subspace $V$ of $\mathbb{R}^N$? Use Fourier analysis:

- Signals of length $N$ decompose into Fourier basis:
  \[ \mathbb{C}^N = \bigoplus_{k \in \mathbb{Z}_N} \mathbb{C}e_k \]
  
  where
  \[ e_k(x) = e^{2ik\pi x/N} . \]

- Main idea: Find subspace of the form
  \[ V = \bigoplus_{k \in K} \mathbb{C}e_k \]
  
  where $K$ is small. $V = \text{vectors with sparse Fourier transform}$
Sparse sums-of-squares

How to look for subspace $V$ of $\mathbb{R}^N$? Use Fourier analysis:

- Signals of length $N$ decompose into Fourier basis:

\[
\mathbb{C}^N = \bigoplus_{k \in \mathbb{Z}_N} \mathbb{C}e_k
\]

where

\[
e_k(x) = e^{2ik\pi x/N}.
\]

- Main idea: Find subspace of the form

\[
V = \bigoplus_{k \in K} \mathbb{C}e_k
\]

where $K$ is small. $V = \text{vectors with sparse Fourier transform}$

- Fourier-analytic question: Find a small $K$ such that any nonnegative bandlimited vector with bandwidth $d$ has a sum-of-squares representation with functions $f_j \in V$
Result

Using graph-theoretic tools we show:

**Theorem**

If $d$ divides $N$ then $\text{rank}_{\text{psd}}(TC_{N,d}) \leq 3d \log(N/d)$.

$\Rightarrow$ SDP factorization of size $\leq 3d \log(N/d)$ of the slack matrix of $TC_{N,d}$. 

Using graph-theoretic tools we show:

**Theorem**

If $d$ divides $N$ then $\text{rank}_{\text{psd}}(TC_{N,d}) \leq 3d \log(N/d)$.

$\Rightarrow$ SDP factorization of size $\leq 3d \log(N/d)$ of the slack matrix of $TC_{N,d}$.

Gap is obtained in regime $N = d^2$:

- SDP lift has size $3d \log(d)$
- $\text{rank}_{\text{LP}}(TC_{d^2,d}) \geq d^2$ [Fiorini et al.]
One ingredient of proof: sparse psd matrices

Positive semidefinite matrices with chordal sparsity

**Theorem** (Grone, Johnson, Sá, Wolkowicz, 1984; Griewank, Toint 1984)

If $Q \succeq 0$ and sparse w.r.t. chordal graph $\Gamma$, then $Q$ decomposes as sum of psd matrices each supported on a maximal clique of $\Gamma$. 

\[
\begin{align*}
\begin{array}{ccc}
\begin{array}{cccc}
\cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \\
\end{array} & = & \\
\begin{array}{cc}
\begin{array}{c}
\cdot \\
\end{array} & \\
\end{array} & + & \\
\begin{array}{cc}
\begin{array}{c}
\cdot \\
\end{array} & \\
\end{array} & + & \cdots
\end{align*}
\]
Conclusion

- Semidefinite representations of convex sets
- Connection with matrix factorization and sums of squares
- Linear programming vs. semidefinite programming lifts for polytopes

For more information:
- [Fawzi, Gouveia, Parrilo, Robinson, Thomas, Positive semidefinite rank, Math. Prog., 2015]
- [Fawzi, Saunderson, Parrilo, Sparse sums of squares on finite abelian groups and improved semidefinite lifts, Math. Prog., 2016]
Conclusion

- Semidefinite representations of convex sets
- Connection with matrix factorization and sums of squares
- Linear programming vs. semidefinite programming lifts for polytopes

For more information:
- [Fawzi, Gouveia, Parrilo, Robinson, Thomas, Positive semidefinite rank, Math. Prog., 2015]
- [Fawzi, Saunderson, Parrilo, Sparse sums of squares on finite abelian groups and improved semidefinite lifts, Math. Prog., 2016]

Thank you!