On the discretization of some nonlinear Fokker-Planck-Kolmogorov equations and applications

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Plan

Introduction

The numerical scheme

Applications
Plan

Introduction

The numerical scheme

Applications
Introduction

We consider the following non-linear \((FPK)\) equation

\[
\partial_t m - \frac{1}{2} \sum_{i,j} \partial^2_{i,j} (a_{i,j}[m]m) + \text{div} \left( b[m](x,t)m \right) = 0,
\]

\[
m(0) = m_0.
\]

\(\Rightarrow\) \(m_0 \in \mathcal{P}_2(\mathbb{R}^d)\).

\(\Rightarrow\) \(b : C([0,T];\mathcal{P}_1(\mathbb{R}^d)) \times \mathbb{R}^d \times [0,T] \rightarrow \mathbb{R}^d\).

\(\Rightarrow\) \(a_{i,j} = \sum_k \sigma_{ik}\sigma_{jk}\) where \(\sigma_{i,j} : C([0,T];\mathcal{P}_1(\mathbb{R}^d)) \times \mathbb{R}^d \times [0,T] \rightarrow \mathbb{R}\)

\(\Rightarrow\) \(m \in C([0,T];\mathcal{P}_1(\mathbb{R}^d))\) solves \((FPK)\) if \(\forall \varphi \in C_0^\infty(\mathbb{R}^d)\)

\[
\int_{\mathbb{R}^d} \varphi(x)dm(t)(x) = \int_{\mathbb{R}^d} \varphi(x)dm_0(x) + \int_0^t \int_{\mathbb{R}^d} [b[m](x,s) \cdot \nabla \varphi(x)] dm(s)(x)ds
\]
\[
+ \int_0^t \int_{\mathbb{R}^d} \left[ \frac{1}{2} \sum_{i,j} a_{i,j}[m](x,s) \partial^2_{i,j} \varphi(x) \right] dm(s)(x)ds.
\]
Formally, a solution is given by
\[ t \in [0, T] \rightarrow m(t) := \text{Law}(X(t)) \in C([0, T]; \mathcal{P}_1(\mathbb{R}^d)), \]
where \( X \) solves the McKean-Vlasov SDE
\[ dX(t) = b[m](X(t), t)dt + \sigma[m](X(t), t)dW(t), \quad X(0) = X_0, \]
\( W \) is a BM in \( \mathbb{R}^r \) and \( X_0 \) is independent of \( W \) and \( \text{Law}(X_0) = m_0 \).

Some references in the linear case (\( b \) et \( \sigma \) independent of \( m \)):
- \( \sigma = 0 \) and \( b \) rough: DiPerna-Lions ’89, Ambrosio ’04, Popaud-Rascle ’07, ..., \( \sigma \neq 0 \), \( b \) roughs: LeBris-Lions ’08, Figalli ’08, Ambrosio-Figalli ’08, monograph by Bogachev, Krylov, Rockner and Shaposhnikov ’15, etc...

Some references in the nonlinear case
- \( \sigma = 0 \): Bogachev, Röckner and Shaposhnikov ’09.
- \( \sigma \neq 0 \): Manita and Shaposhnikov ’12 and ’13.
Plan

Introduction

The numerical scheme

Applications
Numerical approximation

The deterministic case $\sigma = 0$

In order to introduce the scheme assume that $\sigma = 0$ and $b$ independent of $m$. Under suitable assumptions on $b$ we have

$$m(t) = \Phi(0, t, \cdot) \llcorner m_0 = \Phi(s, t, \cdot) \llcorner m(s)$$

where, for every $x \in \mathbb{R}^d$, $0 \leq s \leq t$, $\Phi(s, t, x) := X^{x, s}(t)$ where $X^{x, s}$ solves

$$\dot{X}(s') = b(X(s'), s') \quad s' \in ]s, T[, \quad X(s) = x.$$ 

Time discretization:

Let $\Delta t = T/N$ and set $t_k := k\Delta t$ ($k = 0, \ldots, N$).
The representation formula for $m$ induce the approximation

$$m_{k+1} = \Phi_k(\cdot) m_k \quad (k = 0, \ldots, N - 1),$$

where

$$\Phi_k(x) := x + \Delta t b(x, t_k).$$

Space discretization:

1. Let $\Delta x > 0$ be a space-step and

   $$G_{\Delta x} := \{x_i = i\Delta x ; i \in \mathbb{Z}^d\}.$$  

2. Consider a $\mathbb{P}_1$-basis $\{\beta_i ; i \in \mathbb{Z}^d\}$

\[\text{Fonction } \beta_i \text{ en dimension } d = 1.\]
Letting $\Phi_{i,k} := \Phi_k(x_i) = x_i + \Delta tb(x_i, t_k)$, the representation formula suggest the scheme \(^1\)

$$m_k = \sum_{i \in \mathbb{Z}^d} m_{i,k} \delta_{x_i}$$

with

$$m_{i,0} = m_0(E_i),$$

$$m_{i,k+1} = \sum_{j \in \mathbb{Z}^d} \beta_i(\Phi_{j,k}) m_{j,k}$$

**Stochastic case $\sigma \neq 0$**

- If $b$ and $\sigma$ are independent of $m$ we have the representation formula:

  $$\forall A \in \mathcal{B}(\mathbb{R}^d)$$

  $$m(t)(A) = \mathbb{E}(\Phi(0, t, \cdot, \omega)\#m_0(A)) = \mathbb{E}(\Phi(s, t, \cdot, \omega)\#m(s)(A)),$$

\(^1\)See also Piccoli-Tosin '11 and Tosin-Frasca '11.
where $\Phi(s, t, x, \omega) := X^{x, s}(t, \omega)$ solves

\[
\begin{align*}
    dX(s') &= b(X(s'), s') ds' + \sigma(X(s'), s') dW(s'), \quad s' \in ]s, T[ \\
    X(s) &= x.
\end{align*}
\]

If we discretize the BM $W(\cdot)$ by a random walk and we define

\[
\Phi_{j,k}^{\ell, \pm} = x_j + \Delta t b(x_j, t_k) \pm \sqrt{r \Delta t} \sigma_{\ell}(x_j, t_k),
\]

the representation formula suggest the discretization

\[
m_k = \sum_{i \in \mathbb{Z}^d} m_{i,k} \delta_{x_i}
\]

where

\[
m_{i,0} = m_0(E_i),
\]

\[
m_{i,k+1} = \frac{1}{2r} \sum_{\ell=1}^r \sum_{j \in \mathbb{Z}^d} \left[ \beta_i(\Phi_{j,k}^{\ell, +}) + \beta_i(\Phi_{j,k}^{\ell, -}) \right] m_{j,k}
\]
The numerical scheme

It is easy to see that \( \{m_{i,k} ; i \in \mathbb{Z}^d, k = 0, \ldots, N\} \) are the marginal laws of a discrete-time, discrete-state space Markov chain \( X_0, X_1, \ldots, X_N \) with initial law \( m_{\cdot,0} \) and transition probabilities

\[
p_{j,i}^k := \frac{1}{2r} \sum_{\ell=1}^r \left[ \beta_i(\Phi_{j,k}^{\ell,\ell}) + \beta_i(\Phi_{j,k}^{\ell,\ell}) \right]
\]

In the non-linear case, a fixed-point argument suggests the approximation

\[
m_{i,0} = \frac{1}{(\Delta x)^d} \int_{E_i} m_0(x) dx,
\]

\[
m_{i,k+1} = \frac{1}{2r} \sum_{\ell=1}^r \sum_{j \in \mathbb{Z}^d} \left[ \beta_i(\Phi_{j,k}^{\ell,\ell}[m]) + \beta_i(\Phi_{j,k}^{\ell,\ell}[m]) \right] m_{j,k},
\]

where

\[
\Phi_{j,k}^{\ell,\ell}[m] = x_j + \Delta t b[m](x_j, t_k) \pm \sqrt{r \Delta t} \sigma_{\ell}[m](x_j, t_k).
\]

If \( b[m](x, t) \) and \( \sigma[m](x, t) \) depend on \( \{m(s) ; 0 \leq s \leq t\} \), the scheme is explicit. Otherwise, the scheme is implicit.
Convergence

We assume that

\((H)\)

(i) [Continuity] The maps \(b\) and \(\sigma\) are continuous.

(ii) [Linear growth] There exists \(C > 0\) such that

\[
|b[\mu](x, t)| + |\sigma[\mu](x, t)| \leq C(1 + |x|) \quad \forall \mu \in C([0, T]; \mathcal{P}_1(\mathbb{R}^d)), \quad t \in [0, T].
\]

▶ Under \((H)\) the scheme admits at least one solution.

▶ Let \(m^{\Delta x, \Delta t}\) be a solution of the scheme and extend it to \(C([0, T]; \mathcal{P}_1(\mathbb{R}^d))\) by

\[
m^{\Delta x, \Delta t}(t) := \left(\frac{t - t_k}{\Delta t}\right) m^{\Delta x, \Delta t}_{k+1} + \left(\frac{t_{k+1} - t}{\Delta t}\right) m^{\Delta x, \Delta t}_k \quad \text{if} \quad t \in [t_k, t_{k+1}].
\]
The numerical scheme

- Let \((\Delta_n x, \Delta_n t) \to (0, 0)\) be such that \(\frac{(\Delta_n x)^2}{\Delta_n t} \to 0\) and let \(m^n := m^{\Delta_n x, \Delta_n t}\).

- The interpretation of the scheme in terms of Markov chains allows us to prove the existence of \(C > 0\), independent of \(n\), such that

\[
\int_{\mathbb{R}^d} |x|^2 dm^n(s) \leq c \quad \forall \ s \in [0, T].
\]

- These estimates imply that \(\{m^n ; n \in \mathbb{N}\} \subseteq C([0, T]; \mathcal{P}_1(\mathbb{R}^d))\) is relatively compact.

**Theorem (Carlini-S’17)**

*Under assumption \((H)\), every limit point \(m \in C([0, T]; \mathcal{P}_1(\mathbb{R}^d))\) of \(m^n\) solves \((FPK)\). In particular, under assumption \((H)\), \((FKP)\) admits at least one solution.*
The previous result provides a Peano type existence result for the \((FPK)\) equation under rather general assumptions.

In some cases, as we will see later, we do not have an explicit expression for the coefficients and so we have to approximate them.

Let \(b_n\) and \(\sigma_n\) have linear growth, independent of \(n\), and such that for every sequence \(\mu_n \to \mu\) in \(C([0,T];\mathcal{P}_1(\mathbb{R}^d))\) we have that

\[
\begin{align*}
    b^n[\mu_n](\cdot,\cdot) & \to b[\mu](\cdot,\cdot) \quad \text{and} \quad \\
    \sigma^n[\mu_n](\cdot,\cdot) & \to \sigma[\mu](\cdot,\cdot)
\end{align*}
\]

uniformly in compact subsets of \(\mathbb{R}^d \times [0,T]\).

Construct the scheme for \((\Delta_n x, \Delta_n t)\) using \(b^n\) and \(\sigma^n\).

**Theorem (Carlini-S’17)**

Under the above assumptions, every limit point \(m \in C([0,T];\mathcal{P}_1(\mathbb{R}^d))\) of \(m^n\) (there exists at least one) solves \((FPK)\).
Plan

Introduction

The numerical scheme

Applications
Application in MFGs

We consider the MFG system (Lasry-Lions ’07)

\[ \begin{align*}
-\partial_t v - \frac{\sigma^2}{2} \Delta v + \frac{1}{2} |\nabla v|^2 &= F(x, m(t)), \\
\partial_t m - \frac{\sigma^2}{2} \Delta m - \text{div}(\nabla v m) &= 0, \\
v(x, T) &= G(x, m(t)) \quad \text{for } x \in \mathbb{R}^d, \quad m(0) = m_0(\cdot),
\end{align*} \]  

(MFG)

where \( \sigma \neq 0 \). This system corresponds to the non-linear (FPK) equation

\[ \partial_t m - \frac{\sigma^2}{2} \Delta m - \text{div}(\nabla v[m] m) = 0 \quad \text{in } \mathbb{R}^d \times (0, T), \quad m(0) = m_0, \]

where

\[ v[m](x, t) := \inf_{\alpha} \mathbb{E} \left( \int_t^T \left[ \frac{1}{2} |\alpha(s)|^2 + F(X^{x,t,\alpha}(s), m(s)) \right] ds + G(X^{x,t,\alpha}(T), m(T)) \right) \]

and \( X^{x,t,\alpha} \) solves

\[ dX(s) = \alpha(s) ds + \sigma dW(s) \quad s \in (t, T), \quad X(t) = x. \]
Application in a Hughes type model

We consider the \((FPK)\) equation

\[
\partial_t m - \frac{\sigma^2}{2} \Delta m - \text{div}(\nabla v[m] m) = 0 \quad \text{in} \quad \mathbb{R}^d \times (0, T), \quad m(0) = m_0,
\]

where

\[
v[m](x, t) := \inf_{\alpha} \mathbb{E} \left( \int_t^T \left[ \frac{1}{2} |\alpha(s)|^2 + F(X^{x,t,\alpha}(s), m(t)) \right] \, ds + G(X^{x,t,\alpha}(T), m(t)) \right),
\]

and \(X^{x,t,\alpha}\) solves

\[
dX(s) = \alpha(s) \, ds + \sigma \, dW(s) \quad s \in (t, T), \quad X(t) = x.
\]

- We obtain a convergent explicit scheme in general dimensions.
Numerical simulations

- We let

\[ m_0(x) = e^{-x^2/0.2}, \quad F(x, m) = d(x, \mathcal{P})^2 V_\delta(x, m), \quad G \equiv F, \]

where \( d(\cdot, \mathcal{P}) \) is the distance function to \( \mathcal{P} := [-2.5, -2] \cup [1, 1.5] \).

- Numerical result for the explicit model in 1D.
Numerical result for the corresponding MFG model in 1D.
Systems of Fokker-Planck-Kolmogorov equations

We also extend our scheme for systems of FPK equations

\[ \partial_t m^\ell - \frac{1}{2} \sum_{i,j} \partial_{i,j}^2 \left( a^{\ell}_{i,j}[m] m^\ell \right) + \text{div} \left( b^{\ell}[m] m^\ell \right) = 0, \quad \text{in} \; \mathbb{R}^{d_\ell} \times (0,T), \]

\[ m^\ell(0) = \bar{m}_0 \quad \text{in} \; \mathbb{R}^{d_\ell}, \]

where \( \ell = 1, \ldots, M \) and \( m = (m^1, \ldots, m^M) \).

- We suppose also uniform linear growth for the coefficients.
- It is easy to see that solutions of the previous FPK systems can be obtained as by taking the marginal of the solutions of a suitable FPK equation in the product space.
- We can easily extend the our previous scheme to this case.
- The proof of convergence is similar.
- Several applications: Population dynamics, multi-population MFGs, etc.
Example

We consider the FPK system (see Achdou-Bardi-Cirant)

\[
\begin{align*}
-\partial_t v^1 - \nu \Delta v^1 + \frac{1}{2} |\nabla v^1|^2 &= V(m^1, m^2), \\
-\partial_t v^2 - \nu \Delta v^2 + \frac{1}{2} |\nabla v^2|^2 &= V(m^2, m^1), \\
v^1(\cdot, T) &= 0, \quad v^2(\cdot, T) = 0,
\end{align*}
\]

\[\begin{align*}
\partial_t m^1 - \nu \Delta m^1 - \text{div}(\nabla v^1 m^1) &= 0, \\
\partial_t m^2 - \nu \Delta m^2 - \text{div}(\nabla v^2 m^2) &= 0,
\end{align*}\]

\[m^1(\cdot, 0) = \bar{m}_0(\cdot), \quad m^2(\cdot, 0) = \bar{m}_0(\cdot).\]

where

\[V(m^1, m^2) = \left(\frac{m^1}{m^1 + m^2} - 0.7\right)^- + (m^1 + m^2 - 8)^+,
\]  

(1)
On the discretization of some nonlinear Fokker-Planck-Kolmogorov equations and applications

Applications

Perspectives

▶ Approximations by general Markov chains.
▶ Higher order schemes.
▶ Boundary conditions.
▶ Ergodic case.

References


