On the discretization of some nonlinear Fokker-Planck-Kolmogorov equations and applications

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Introduction

We consider the following non-linear (FPK) equation

$$\begin{aligned} \partial_t m - \frac{1}{2}\sum_{i,j} \partial^2_{i,j}(a_{i,j}[m]m) + \text{div}\,(b[m](x,t)m) &= 0, \\ m(0) &= m_0. \end{aligned} \qquad (FPK)$$

- $m_0 \in \mathcal{P}_2(\mathbb{R}^d)$.
- $b : C([0,T];\mathcal{P}_1(\mathbb{R}^d)) \times \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$.
- $a_{i,j} = \sum_k \sigma_{ik}\sigma_{jk}$ where $\sigma_{i,j} : C([0,T];\mathcal{P}_1(\mathbb{R}^d)) \times \mathbb{R}^d \times [0,T] \to \mathbb{R}$
- $m \in C([0,T];\mathcal{P}_1(\mathbb{R}^d))$ solves (FPK) if $\forall\, \varphi \in C_0^\infty(\mathbb{R}^d)$

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x)\mathrm{d}m(t)(x) &= \int_{\mathbb{R}^d} \varphi(x)\mathrm{d}m_0(x) \\ &+ \int_0^t \int_{\mathbb{R}^d} \left[b[m](x,s)\cdot\nabla\varphi(x)\right]\mathrm{d}m(s)(x)\mathrm{d}s \\ &+ \int_0^t \int_{\mathbb{R}^d} \left[\frac{1}{2}\sum_{i,j} a_{i,j}[m](x,s)\partial^2_{ij}\varphi(x)\right]\mathrm{d}m(s)(x)\mathrm{d}s \end{aligned}$$

- Formally, a solution is given by $t \in [0,T] \to m(t) := \text{Law}(X(t)) \in C([0,T];\mathcal{P}_1(\mathbb{R}^d))$, where X solves the McKean-Vlasov SDE

$$\mathrm{d}X(t) = b[m](X(t),t)\mathrm{d}t + \sigma[m](X(t),t)\mathrm{d}W(t), \quad X(0) = X_0,$$

  W is a BM in $\mathbb{R}^r$ and X_0 is independent of W and $\text{Law}(X_0) = m_0$.
- Some references in the linear case (b et σ independent of m):
  - $\sigma = 0$ and b rough: DiPerna-Lions '89, Ambrosio '04, Popaud-Rascle '07,... , $\sigma \neq 0$, b roughs: LeBris-Lions '08, Figalli '08, Ambrosio-Figalli '08, monograph by Bogachev, Krylov, Rockner and Shaposhnikov '15, etc...
- Some references in the nonlinear case
  - $\sigma = 0$: Bogachev, Röckner and Shaposhnikov '09.
    $\sigma \neq 0$: Manita and Shaposhnikov '12 and '13.

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Numerical approximation

The deterministic case $\sigma = 0$

- In order to introduce the scheme assume that $\sigma = 0$ and b independent of m. Under suitable assumptions on b we have

$$m(t) = \Phi(0, t, \cdot)\sharp m_0 = \Phi(s, t, \cdot)\sharp m(s)$$

where, for every $x \in \mathbb{R}^d$, $0 \leq s \leq t$, $\Phi(s, t, x) := X^{x,s}(t)$ where $X^{x,s}$ solves

$$\begin{aligned} \dot{X}(s') &= b(X(s'), s') \quad s' \in]s, T[, \\ X(s) &= x. \end{aligned}$$

Time discretization:

- Let $\Delta t = T/N$ and set $t_k := k\Delta t$ $(k = 0, \ldots, N)$.

- The representation formula for m induce the approximation

$$m_{k+1} = \Phi_k(\cdot)\sharp m_k \quad (k = 0, \dots, N-1),$$

where

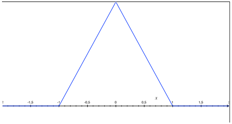
$$\Phi_k(x) := x + \Delta t b(x, t_k).$$

Space discretization:

- Let $\Delta x > 0$ be a space-step and

$$\mathcal{G}_{\Delta x} := \{x_i = i\Delta x \; ; \; i \in \mathbb{Z}^d\}.$$

- Consider a $\mathbb{P}_1$-basis $\{\beta_i \; ; \; i \in \mathbb{Z}^d\}$



Fonction β_i en dimension $d = 1$.

- Letting $\Phi_{i,k} := \Phi_k(x_i) = x_i + \Delta t b(x_i, t_k)$, the representation formula suggest the scheme [1]

$$m_k = \sum_{i \in \mathbb{Z}^d} m_{i,k} \delta_{x_i}$$

with

$$\begin{aligned} m_{i,0} &= m_0(E_i), \\ m_{i,k+1} &= \textstyle\sum_{j \in \mathbb{Z}^d} \beta_i(\Phi_{j,k}) m_{j,k} \end{aligned}$$

Stochastic case $\sigma \neq 0$

- If b and σ are independent of m we have the representation formula: $\forall\, A \in \mathcal{B}(\mathbb{R}^d)$

$$m(t)(A) = \mathbb{E}\left(\Phi(0,t,\cdot,\omega)\sharp m_0(A)\right) = \mathbb{E}\left(\Phi(s,t,\cdot,\omega)\sharp m(s)(A)\right),$$

[1] See also Piccoli-Tosin '11 and Tosin-Frasca '11.

where $\Phi(s,t,x,\omega) := X^{x,s}(t,\omega)$ solves

$$\begin{aligned} \mathrm{d}X(s') &= b(X(s'),s')\mathrm{d}s' + \sigma(X(s'),s')\mathrm{d}W(s'), \; s' \in]s,T[\\ X(s) &= x. \end{aligned}$$

- If we discretize the BM $W(\cdot)$ by a random walk and we define

$$\Phi_{j,k}^{\ell,\pm} = x_j + \Delta t b(x_j,t_k) \pm \sqrt{r\Delta t}\sigma_\ell(x_j,t_k),$$

the representation formula suggest the discretization

$$m_k = \sum_{i\in\mathbb{Z}^d} m_{i,k}\delta_{x_i}$$

where

$$m_{i,0} = m_0(E_i),$$

$$m_{i,k+1} = \tfrac{1}{2r}\textstyle\sum_{\ell=1}^{r}\sum_{j\in\mathbb{Z}^d}\left[\beta_i(\Phi_{j,k}^{\ell,+}) + \beta_i(\Phi_{j,k}^{\ell,-})\right] m_{j,k}$$

- It is easy to see that $\{m_{i,k} \; ; \; i \in \mathbb{Z}^d, \;\; k = 0, \dots, N\}$ are the marginal laws of a discrete-time, discrete-state space Markov chain $X_0, \; X_1, \dots, X_N$ with initial law $m_{\cdot,0}$ and transition probabilities

$$p_{j,i}^{k} := \frac{1}{2r} \sum_{\ell=1}^{r} \left[\beta_i(\Phi_{j,k}^{\ell,+}) + \beta_i(\Phi_{j,k}^{\ell,-}) \right]$$

- In the non-linear case, a fixed-point argument suggests the approximation

$$m_{i,0} = \tfrac{1}{(\Delta x)^d} \int_{E_i} m_0(x) \mathrm{d}x,$$
$$m_{i,k+1} = \tfrac{1}{2r} \sum_{\ell=1}^{r} \sum_{j \in \mathbb{Z}^d} \left[\beta_i(\Phi_{j,k}^{\ell,+}[m]) + \beta_i(\Phi_{j,k}^{\ell,-}[m]) \right] m_{j,k},$$

where

$$\Phi_{j,k}^{\ell,\pm}[m] = x_j + \Delta t b[m](x_j, t_k) \pm \sqrt{r \Delta t} \sigma_\ell[m](x_j, t_k).$$

- If $b[m](x,t)$ and $\sigma[m](x,t)$ depend on $\{m(s) \; ; \; 0 \leq s \leq t\}$, the scheme is explicit. Otherwise, the scheme is implicit.

Convergence

We assume that

(H)

(i) [Continuity] The maps b and σ are continuous.

(ii) [Linear growth] There exists $C > 0$ such that

$$|b[\mu](x,t)|+|\sigma[\mu](x,t)| \leq C(1+|x|) \;\; \forall\, \mu \in C([0,T];\mathcal{P}_1(\mathbb{R}^d)), \;\; t \in [0,T].$$

- Under **(H)** the scheme admits at least one solution.
- Let $m^{\Delta x,\Delta t}$ be a solution of the scheme and extend it to $C([0,T];\mathcal{P}_1(\mathbb{R}^d))$ by

$$m^{\Delta x,\Delta t}(t) := \left(\frac{t-t_k}{\Delta t}\right) m_{k+1}^{\Delta x,\Delta t} + \left(\frac{t_{k+1}-t}{\Delta t}\right) m_k^{\Delta x,\Delta t} \;\; \text{if } t \in [t_k, t_{k+1}[.$$

- Let $(\Delta_n x, \Delta_n t) \to (0,0)$ be such that $\frac{(\Delta_n x)^2}{\Delta_n t} \to 0$ and let $m^n := m^{\Delta_n x, \Delta_n t}$
- The interpretation of the scheme in terms of Markov chains allows us to prove the existence of $C > 0$, independent of n, such that

$$d_1(m^n(t), m^n(s)) \leq C|t-s|^{\frac{1}{2}} \quad (\leq C|t-s| \text{ if } \sigma = 0),$$
$$\int_{\mathbb{R}^d} |x|^2 \mathrm{d}m^n(s) \leq c \quad \forall\, s \in [0,T].$$

- These estimates imply that $\{m^n \ ; \ n \in \mathbb{N}\} \subseteq C([0,T]; \mathcal{P}_1(\mathbb{R}^d))$ is relatively compact.

Theorem (Carlini-S'17)

Under assumption **(H)**, *every limit point* $m \in C([0,T]; \mathcal{P}_1(\mathbb{R}^d))$ *of* m^n *solves* (FPK). *In particular, under assumption* **(H)**, (FKP) *admits at least one solution.*

- The previous result provides a Peano type existence result for the (FPK) equation under rather general assumptions.
- In some cases, as we will see later, we do not have an explicit expression for the coefficients and so we have to approximate them.
- Let b_n and σ_n have linear growth, independent of n, and such that for every sequence $\mu_n \to \mu$ in $C([0,T];\mathcal{P}_1(\mathbb{R}^d))$ we have that

$$b^n[\mu_n](\cdot,\cdot) \to b[\mu](\cdot,\cdot) \quad \text{and} \quad \sigma^n[\mu_n](\cdot,\cdot) \to \sigma[\mu](\cdot,\cdot)$$

  uniformly in compact subsets of $\mathbb{R}^d \times [0,T]$.
- Construct the scheme for $(\Delta_n x, \Delta_n t)$ using b^n and σ^n.

Theorem (Carlini-S'17)

Under the above assumptions, every limit point $m \in C([0,T];\mathcal{P}_1(\mathbb{R}^d))$ of m^n (there exists at least one) solves (FPK).

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Application in MFGs

We consider the MFG system (Lasry-Lions '07)

$$\begin{aligned} -\partial_t v - \frac{\sigma^2}{2}\Delta v + \frac{1}{2}|\nabla v|^2 &= F(x, m(t)), \\ \partial_t m - \frac{\sigma^2}{2}\Delta m - \text{div}\big(\nabla v m\big) &= 0, \\ v(x,T) = G(x, m(t)) \ \text{ for } x \in \mathbb{R}^d, &\qquad m(0) = m_0(\cdot), \end{aligned} \qquad (MFG)$$

where $\sigma \neq 0$. This system corresponds to the non-linear (FPK) equation

$$\partial_t m - \frac{\sigma^2}{2}\Delta m - \text{div}\big(\nabla v[m] m\big) = 0 \ \text{ in } \ \mathbb{R}^d \times (0,T), \quad m(0) = m_0,$$

where

$$v[m](x,t) := \inf_{\alpha} \mathbb{E}\left(\int_t^T \left[\tfrac{1}{2}|\alpha(s)|^2 + F(X^{x,t,\alpha}(s), m(s))\right] \mathrm{d}s + G(X^{x,t,\alpha}(T), m(T)) \right)$$

and $X^{x,t,\alpha}$ solves

$$\mathrm{d}X(s) = \alpha(s)\mathrm{d}s + \sigma \mathrm{d}W(s) \ \ s \in (t,T), \qquad X(t) = x.$$

Application in a Hughes type model

We consider the (FPK) equation

$$\partial_t m - \frac{\sigma^2}{2}\Delta m - \text{div}\big(\nabla v[m] m\big) = 0 \quad \text{in } \mathbb{R}^d \times (0,T), \quad m(0) = m_0,$$

where

$$v[m](x,t) := \inf_{\alpha} \mathbb{E}\left(\int_t^T \left[\tfrac{1}{2}|\alpha(s)|^2 + F(X^{x,t,\alpha}(s), m(t))\right] \mathrm{d}s + G(X^{x,t,\alpha}(T), m(t))\right),$$

and $X^{x,t,\alpha}$ solves

$$\mathrm{d}X(s) = \alpha(s)\mathrm{d}s + \sigma \mathrm{d}W(s) \;\; s \in (t,T), \qquad X(t) = x.$$

- We obtain a convergent explicit scheme in general dimensions.

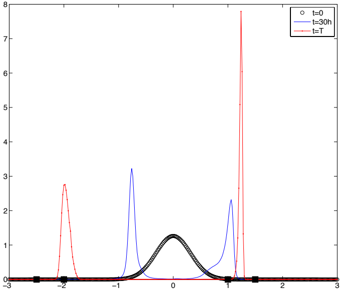
Numerical simulations

- We let

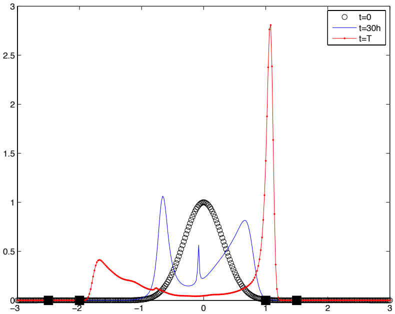
$$m_0(x) = e^{-x^2/0.2}, \quad F(x,m) = d(x,\mathcal{P})^2 V_\delta(x,m), \quad G \equiv F,$$

where $d(\cdot,\mathcal{P})$ is the distance function to $\mathcal{P} := [-2.5,-2] \cup [1,1.5]$.

- Numerical result for the explicit model in 1D.



- Numerical result for the corresponding MFG model in 1D.



Systems of Fokker-Planck-Kolmogorov equations

We also extend our scheme for systems of FPK equations

$$\begin{aligned} \partial_t m^\ell - \tfrac{1}{2}\sum_{i,j}\partial^2_{i,j}\left(a^\ell_{i,j}[m]m^\ell\right) + \mathrm{div}\left(b^\ell[m]m^\ell\right) &= 0, \quad \text{in } \mathbb{R}^{d_\ell}\times(0,T), \\ m^\ell(0) &= \bar{m}^\ell_0 \quad \text{in } \mathbb{R}^{d_\ell}, \end{aligned}$$

where $\ell = 1, \dots, M$ and $m = (m^1, \dots, m^M)$.

- We suppose also uniform linear growth for the coefficients.
- It is easy to see that solutions of the previous FPK systems can be obtained as by taking the marginal of the solutions of a suitable FPK equation in the product space.
- We can easily extend the our previous scheme to this case.
- The proof of convergence is similar.
- Several applications: Population dynamics, multi-population MFGs, etc.

Example

We consider the FPK system (see Achdou-Bardi-Cirant)

$$\begin{cases} -\partial_t v^1 - \nu\Delta v^1 + \frac{1}{2}|\nabla v^1|^2 & = V(m^1, m^2), \\ -\partial_t v^2 - \nu\Delta v^2 + \frac{1}{2}|\nabla v^2|^2 & = V(m^2, m^1), \\ v^1(\cdot, T) = 0, & v^2(\cdot, T) = 0, \\ \partial_t m^1 - \nu\Delta m^1 - \mathrm{div}(\nabla v^1 m^1) & = 0, \\ \partial_t m^2 - \nu\Delta m^2 - \mathrm{div}(\nabla v^2 m^2) & = 0, \\ m^1(\cdot, 0) = \bar{m}_0^1(\cdot), & m^2(\cdot, 0) = \bar{m}_0^2(\cdot). \end{cases} \qquad \text{(MFG)}$$

where

$$V(m^1, m^2) = \left(\frac{m^1}{m^1 + m^2} - 0.7\right)^{-} + (m^1 + m^2 - 8)^{+}, \qquad (1)$$

Perspectives

- Approximations by general Markov chains.
- Higher order schemes.
- Boundary conditions.
- Ergodic case.

References

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