Advertising Competitions in Social Networks

Antonia Maria Masucci[1] and Alonso Silva*[2]

[1]Orange Labs, France

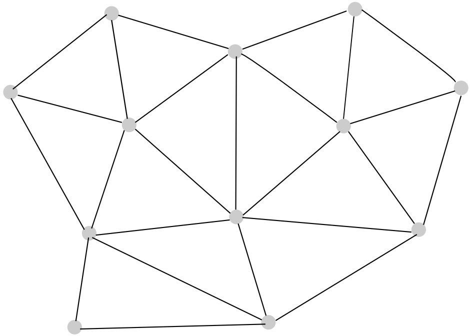
[2]Nokia Bell Labs, France

PGMO Days - November 14, 2017

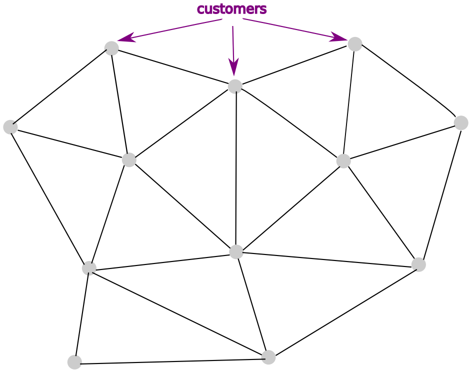
Table of contents

Table of contents

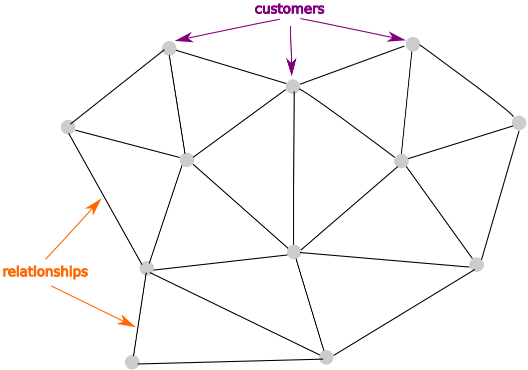
Advertise a new product



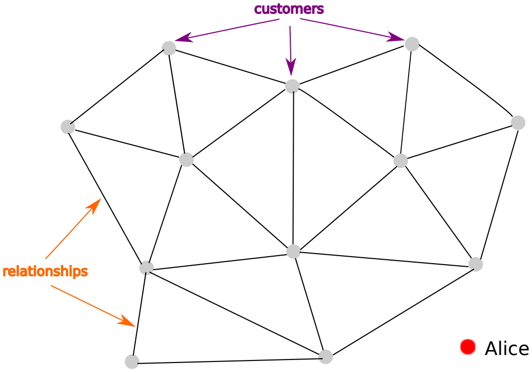
Advertise a new product



Advertise a new product



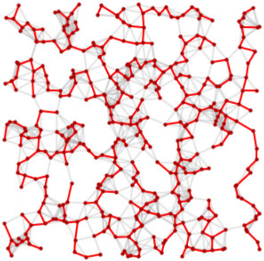
Advertise a new product



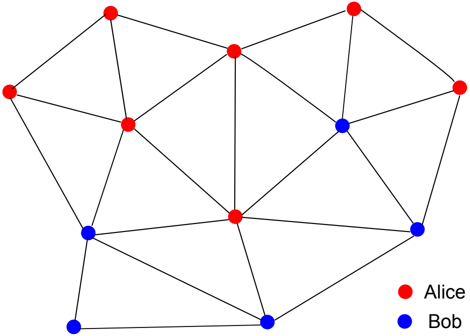
Contamination by one player

- There is a graph
- Contaminate a few nodes
- Let the network work for you
- Main results: $(1 - \frac{1}{e})$ approximations

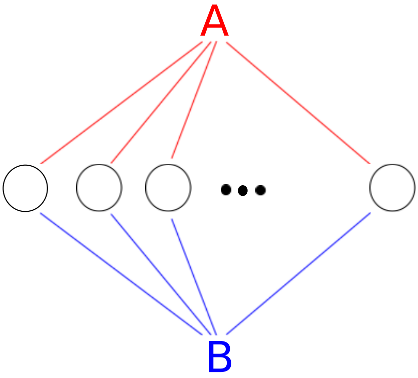
[Kempe, Kleinberg and Tardos, 2003].



Alice vs Bob



Alice vs Bob



Colonel Blotto games

- Battlefields
- Players need to allocate soldiers to each battlefield
- Each battlefield is won by the number of soldiers
- Goal: win more battlefields than the opponents

Colonel Blotto games

- Customers ≡ Battlefields
- Players need to allocate soldiers to each customer
- Each customer is won by the number of soldiers
- Goal: win more customers than the opponents



Colonel Blotto games

- Customers ≡ Battlefields
- Marketing campaigns need to allocate soldiers to each customer
- Each customer is won by the number of soldiers
- Goal: win more customers than the opponents

Colonel Blotto games

- Customers ≡ Battlefields
- Marketing campaigns need to allocate offers or discounts to each customer
- Each customer is won by the offers or discounts
- Goal: win more customers than the opponents

Colonel Blotto games

- Customers ≡ Battlefields
- Marketing campaigns need to allocate offers or discounts to each customer
- Each customer is won by the offers or discounts
- Goal: win more value of the customers than the opponents



Table of contents

Description of the game

- There is a graph of battlefields
- Step one: win battlefields
- Step two: let contagion happen
- Goal: win after step two

Model description

Graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ of customers.

Model description

Graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ of customers. Two players: A and B, and a customer $j \in \mathcal{N}$

Model description

Graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ of customers. Two players: A and B, and a customer $j \in \mathcal{N}$

$$p_{A,j}(x_{A,j}, x_{B,j}) \equiv \text{prob. that player } A \text{ wins customer } j \text{ when player } A \text{ allocates } x_{A,j} \text{ and player } B \text{ allocates } x_{B,j}$$

Model description

Graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ of customers. Two players: A and B, and a customer $j \in \mathcal{N}$

$$\begin{aligned} p_{A,j}(x_{A,j}, x_{B,j}) \equiv & \text{prob. that player } A \text{ wins customer } j \\ & \text{when player } A \text{ allocates } x_{A,j} \\ & \text{and player } B \text{ allocates } x_{B,j} \end{aligned}$$

1. Contest Success Function (CSF):

$$p_{A,j}(x_{A,j}, x_{B,j}) = \begin{cases} 1 & \text{if } x_{A,j} > x_{B,j} \\ 1/2 & \text{if } x_{A,j} = x_{B,j} \\ 0 & \text{if } x_{A,j} < x_{B,j} \end{cases}$$

Model description

Graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ of customers. Two players: A and B, and a customer $j \in \mathcal{N}$

$$p_{A,j}(x_{A,j}, x_{B,j}) \equiv \text{prob. that player } A \text{ wins customer } j \\ \text{when player } A \text{ allocates } x_{A,j} \\ \text{and player } B \text{ allocates } x_{B,j}$$

1. Contest Success Function (CSF):

$$p_{A,j}(x_{A,j}, x_{B,j}) = \begin{cases} 1 & \text{if } x_{A,j} > x_{B,j} \\ 1/2 & \text{if } x_{A,j} = x_{B,j} \\ 0 & \text{if } x_{A,j} < x_{B,j} \end{cases}$$

2. Contest Success Function (CSF'):

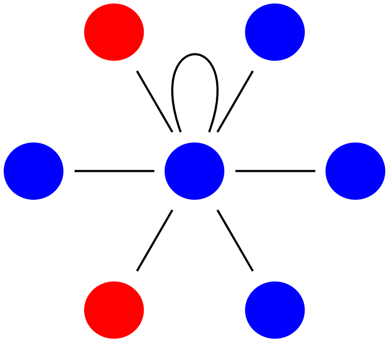
$$p_{A,j}(x_{A,j}, x_{B,j}) = \begin{cases} \frac{x_{A,j}^R}{x_{A,j}^R + x_{B,j}^R} & \text{if } (x_{A,j}, x_{B,j}) \neq (0,0) \\ 1/2 & \text{if } (x_{A,j}, x_{B,j}) = (0,0) \end{cases}$$

Table of contents

Voter model

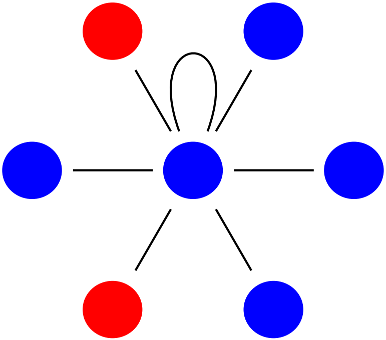
- Given an undirected graph with self-loops
- Starting from an arbitrary initial assignment to the vertices
- At each time $t \geq 1$, each node picks uniformly at random one of its neighbors and adopts its opinion

Voter model



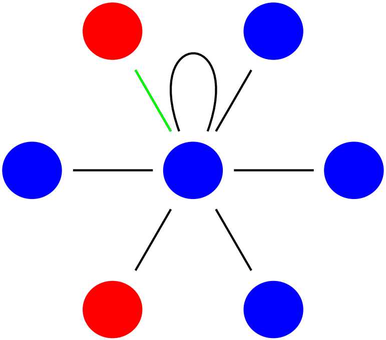
Voter model

- Prob. 5/7 blue
- Prob. 2/7 red

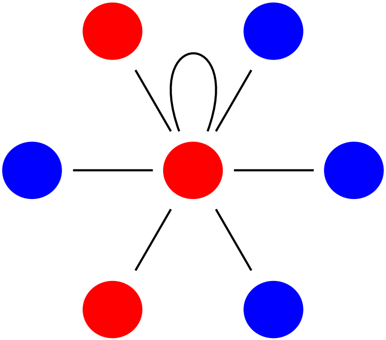


Voter model

- Prob. 5/7 blue
- Prob. 2/7 red

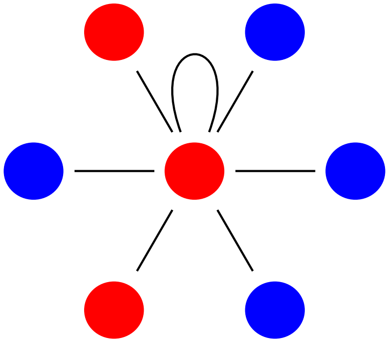


Voter model

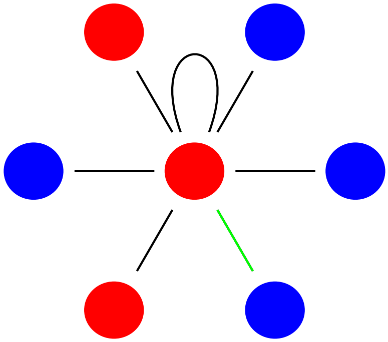


Voter model

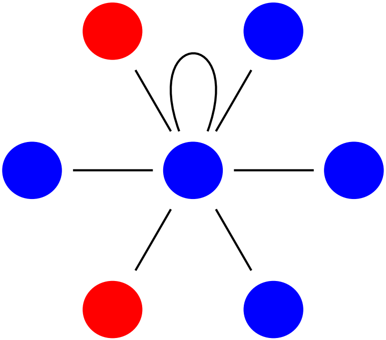
- Prob. 4/7 blue
- Prob. 3/7 red



Voter model



Voter model



Properties of the Voter model

- A person is more likely to change her opinion to the one held by most of her neighbors
- The process is non-monotone (a vertex can be activated and deactivated)
- Consensus is reached with probability one

Optimal strategies

- Reward w_j for winning customer j

Optimal strategies

- Reward w_j for winning customer j
- Simple cases to deal with:

Optimal strategies

- Reward w_j for winning customer j
- Simple cases to deal with:
  - If $\exists w_j, w_j \geq \sum_{k \neq j} w_k$, go for it (pure strategy)!

Optimal strategies

- Reward w_j for winning customer j
- Simple cases to deal with:
  - If $\exists w_j, w_j \geq \sum_{k \neq j} w_k$, go for it (pure strategy)!
  - Never mind all $w_j = 0$

Optimal strategies

- Reward w_j for winning customer j
- Simple cases to deal with:
  - If $\exists w_j, w_j \geq \sum_{k \neq j} w_k$, go for it (pure strategy)!
  - Never mind all $w_j = 0$
- General case, mixed strategy: pick up an allocation so that for customer j, the bid is uniformly distributed in $[0, 2B\frac{w_j}{\sum_k w_k}]$.

Optimal allocation for the voter model

Not so difficult:

- Consider a random walk of length τ
- Choose weights according to the walk distribution
- Special case : $\tau \to \infty \Rightarrow w_j \equiv \text{degree of } j$

Theorem (Masucci and S.)

Given a graph $G = (V, E)$ representing a social network of n potential customers.

Theorem (Masucci and S.)

Given a graph $G = (V, E)$ representing a social network of n potential customers.

- *The symmetric strategic resource allocation problem for target time τ has a solution given by a probability distribution function F^*_τ of $x \in \Delta^{n-1}$, such that each vector coordinate x_i is uniformly distributed on $[0, 2B\sum_{j=1}^{n} M^\tau(i,j)]$ for $i \in \{1, \ldots, n\}$, where*
  - *B is the available budget*
  - *M is the normalized transition matrix*
  - *Δ^{n-1} is the set of available allocations*

Theorem (Masucci and S.)

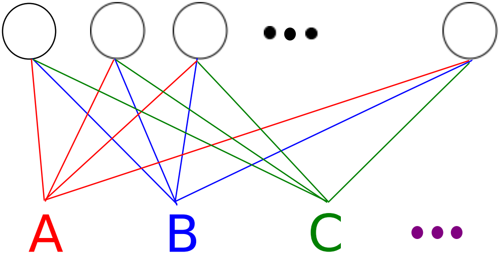
Given a graph $G = (V, E)$ representing a social network of n potential customers.

- *The symmetric strategic resource allocation problem for target time τ has a solution given by a probability distribution function F^*_τ of $x \in \Delta^{n-1}$, such that each vector coordinate x_i is uniformly distributed on $[0, 2B \sum_{j=1}^{n} M^\tau(i, j)]$ for $i \in \{1, \ldots, n\}$, where*
  - *B is the available budget*
  - *M is the normalized transition matrix*
  - *Δ^{n-1} is the set of available allocations*
- *Moreover, the long term case has a solution given by a probability distribution function F^*_∞ of $x \in \Delta^{n-1}$, such that each vector coordinate x_i is uniformly distributed on $[0, Bd_i/|E|]$ for $i \in \{1, \ldots, n\}$, where*
  - *B is the available budget*
  - *d_i is the degree of potential customer i*
  - *$|E|$ is the total number of edges of the graph*
  - *Δ^{n-1} is the set of available allocations*

- "Strategic Resource Allocation for Competitive Influence in Social Networks," Masucci and S., Proc. of the *52nd Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, Illinois, IL, USA, October 1-3, 2014.
- "Defensive Resource Allocation in Social Networks," Masucci and S., Proc. of the *54th IEEE Conference on Decision and Control (CDC)*, Osaka, Japan, December 15-18, 2015.

Table of contents

Alice vs Bob vs Carol vs ...



Pairwise competitions?

Pairwise competitions?

- 3 marketing campaigns, A, B, C, with budget equal to 10

Pairwise competitions?

- 3 marketing campaigns, A, B, C, with budget equal to 10
- 5 customers

Pairwise competitions?

- 3 marketing campaigns, A, B, C, with budget equal to 10
- 5 customers
- Consider the allocations
  - $\mathbf{x_A} = (2, 2, 2, 2, 2)$
  - $\mathbf{x_B} = (0, 0, 0, 5, 5)$
  - $\mathbf{x_C} = (5, 5, 0, 0, 0)$

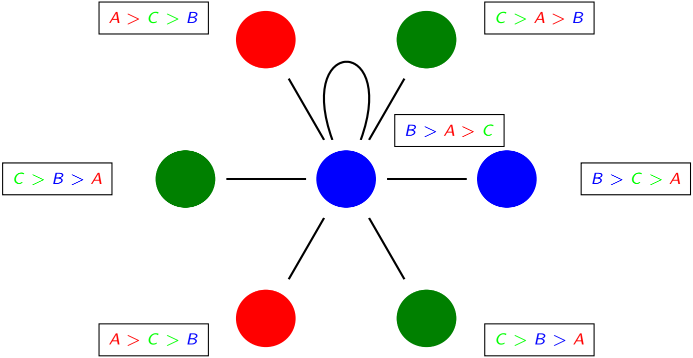
Pairwise competitions?

- 3 marketing campaigns, A, B, C, with budget equal to 10
- 5 customers
- Consider the allocations
  - $\mathbf{x_A} = (2, 2, 2, 2, 2)$
  - $\mathbf{x_B} = (0, 0, 0, 5, 5)$
  - $\mathbf{x_C} = (5, 5, 0, 0, 0)$
- Pairwise competition
  - A gets 3 out of 5 customers against B
  - A gets 3 out of 5 customers againts C

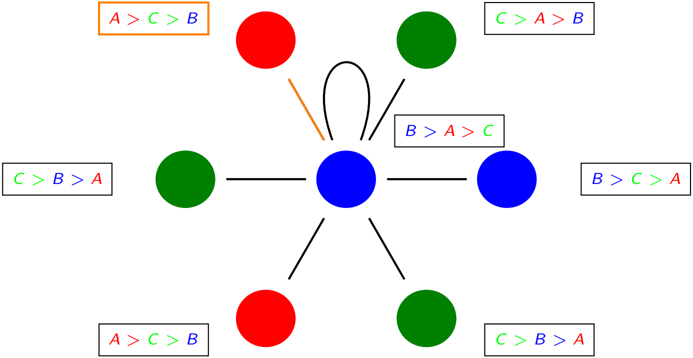
Pairwise competitions?

- 3 marketing campaigns, A, B, C, with budget equal to 10
- 5 customers
- Consider the allocations
  - $\mathbf{x_A} = (2, 2, 2, 2, 2)$
  - $\mathbf{x_B} = (0, 0, 0, 5, 5)$
  - $\mathbf{x_C} = (5, 5, 0, 0, 0)$
- Pairwise competition
  - A gets 3 out of 5 customers against B
  - A gets 3 out of 5 customers againts C
- Each customer chooses one product
  - 1 customer for A
  - 2 customers for B
  - 2 customers for C

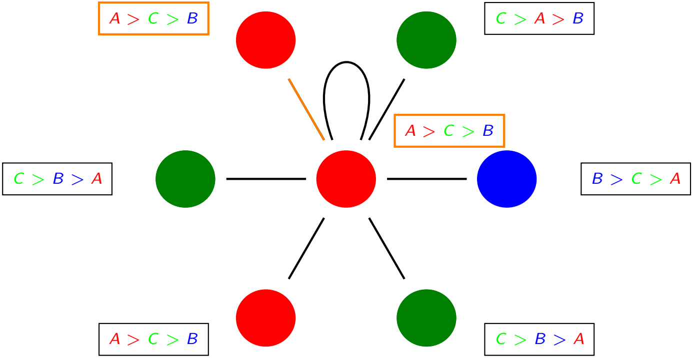
Voter model



Voter model



Voter model



- $\mathcal{K} = \{1, 2, \ldots, K\}$: marketing campaigns
- $\mathcal{V} = \{1, 2, \ldots, N\}$: potential costumers

- $\mathcal{K} = \{1, 2, \dots, K\}$: marketing campaigns
- $\mathcal{V} = \{1, 2, \dots, N\}$: potential costumers

- Assumptions:
  - B: budget of marketing campaigns
  - $\mathbf{s} = (s_1, s_2, \dots, s_K)$: normalized scores assigned to marketing campaigns (w.l.o.g. $s_1 \geq s_2 \geq \dots \geq s_K = 0$ and such that $\sum_{k=1}^{K} s_k = 1$)

- $\mathcal{K} = \{1, 2, \ldots, K\}$: marketing campaigns
- $\mathcal{V} = \{1, 2, \ldots, N\}$: potential costumers

- Assumptions:
  - B: budget of marketing campaigns
  - $\mathbf{s} = (s_1, s_2, \ldots, s_K)$: normalized scores assigned to marketing campaigns (w.l.o.g. $s_1 \geq s_2 \geq \ldots \geq s_K = 0$ and such that $\sum_{k=1}^{K} s_k = 1$)
- $w_n (\leq U)$: intrinsic value of potential customer n
- $w_n \mathbf{s} = (w_n s_1, w_n s_2, \ldots, w_n s_K)$: distributed value across campaigns

- $\mathcal{K} = \{1, 2, \ldots, K\}$: marketing campaigns
- $\mathcal{V} = \{1, 2, \ldots, N\}$: potential costumers

- Assumptions:
  - B: budget of marketing campaigns
  - $\mathbf{s} = (s_1, s_2, \ldots, s_K)$: normalized scores assigned to marketing campaigns (w.l.o.g. $s_1 \geq s_2 \geq \ldots \geq s_K = 0$ and such that $\sum_{k=1}^{K} s_k = 1$)
- $w_n (\leq U)$: intrinsic value of potential customer n
- $w_n \mathbf{s} = (w_n s_1, w_n s_2, \ldots, w_n s_K)$: distributed value across campaigns
- v_n: network value of potential customer $n \rightarrow$ to be determined
- $B v_n / \sum_{j \in \mathcal{V}} v_j$: mean of each marketing campaign's offer distribution for potential customer n

Intrinsic payoff function for marketing campaign k:

$$\pi_k^{\mathrm{INT}} := \sum_{n \in \mathcal{V}} w_n s_{r_n(k)}$$

$s_{r_n(k)}$:= score of the ranking given to campaign k by potential customer n

Intrinsic payoff function for marketing campaign k:

$$\pi_k^{\mathrm{INT}} := \sum_{n \in \mathcal{V}} w_n s_{r_n(k)}$$

$s_{r_n(k)}$:= score of the ranking given to campaign k by potential customer n

Network payoff function for marketing campaign k:

$$\pi_k^t := \sum_{n \in \mathcal{V}} w_n s_{r_n^t(k)}$$

$s_{r_n^t(k)}$:= score of the ranking given to campaign k by potential customer n at time t

Intrinsic payoff function for marketing campaign k:

$$\pi_k^{\mathrm{INT}} := \sum_{n \in \mathcal{V}} w_n s_{r_n(k)}$$

$s_{r_n(k)}$:= score of the ranking given to campaign k by potential customer n

Network payoff function for marketing campaign k:

$$\pi_k^t := \sum_{n \in \mathcal{V}} w_n s_{r_n^t(k)}$$

$s_{r_n^t(k)}$:= score of the ranking given to campaign k by potential customer n at time t

Note that

$$\boxed{\pi_k^{\mathrm{INT}} = \pi_k^0.}$$

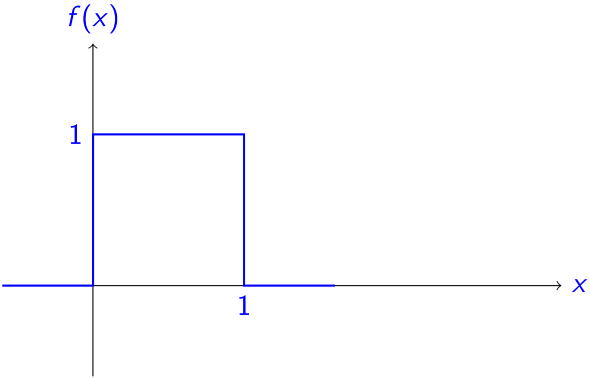
The expected network payoff for marketing campaign k at target time τ is given by

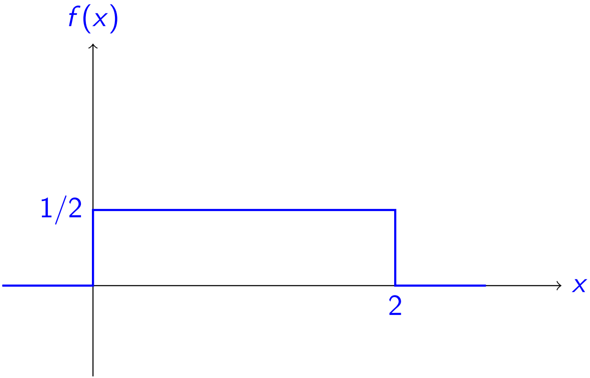
$$\pi_k^\tau = \sum_{n\in\mathcal{V}} w_n s_{r_n^\tau(k)} = \sum_{j\in\mathcal{V}} \sum_{j'\in\mathcal{V}} w_j M^\tau(j,j') s_{r_{j'}(k)}$$

where

$$\boxed{v_{j'} = \sum_{j\in\mathcal{N}} w_j M^\tau(j,j')}$$

corresponds to the network value of potential customer j' at target time τ.


$f(x)$
1
1
x


f(x)
1/2
2
x

Scale family cumulative distributions

Given a cdf function F with bounded support $I = [a, b]$, we define its general cdf G with nonnegative scaling factor v as follows:

$$G(x) := \begin{cases} 0 & \text{for } x \leq va, \\ F(x/v) & \text{for } va \leq x \leq vb, \\ 1 & \text{for } x \geq vb. \end{cases}$$

We call F the *representative* of the scale family cdf.

Marketing campaign evaluation

- $F_n(x)$: cumulative offer distribution for potential customer n
- $P(j, q)$: probability that exactly $j-1$ of the $K-1$ competing marketing campaigns will offer more than x given that each other marketing campaign has an independent probability $(1-q)$ of offering more than x

$$P(j, q) = \binom{K-1}{j-1} q^{K-j}(1-q)^{j-1}.$$

- If marketing campaign k offers x to potential customer n, then the expected value that this potential customer will give to this marketing campaign is $R_n(F_n(x))$ where

$$R_n(q) = v_n \sum_{j=1}^{K} P(j, q) s_j.$$

Unique symmetric equilibrium

Theorem (Masucci and S.)

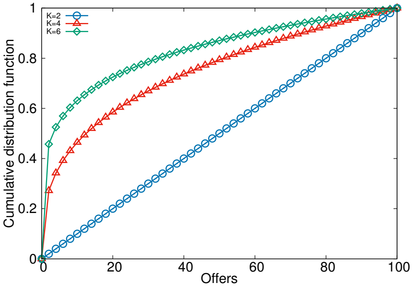
In a K-marketing campaign competition there is a unique symmetric equilibrium: each marketing campaign chooses to generate offers according to a scale family cumulative distribution with scale parameter v_n for potential customer n, which has a cumulative distribution $F(\cdot)$ that satisfies the equation

$$x = R_n(F_n(x))/(V/KB), \quad \forall x \in [0, s_1 KBv_n/V].$$

Simulations 1/2

Winner-takes-all ($\mathbf{s} = (1, 0, 0, \ldots, 0)$) symmetric equilibrium offer distributions

- budget of \$1 000,
- $K = 2$ and 20 customers of average value \$1 000;
- $K = 4$ and 40 customers of average value \$500; and
- $K = 6$ and 60 customers of average value \$333

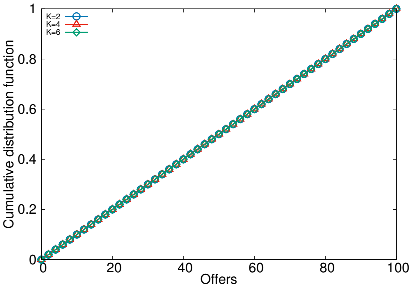


→ For two marketing campaigns: offers are uniformly random.
→ For more, offers are skewed offering less than average to most potential customers and much more for a reduced number.

Simulations 2/2

Borda ($\mathbf{s} = \left(\frac{(K-1)}{C_B}, \frac{(K-2)}{C_B}, \frac{(K-3)}{C_B}, \ldots, 0\right)$) symmetric equilibrium offer distribution

- budget of \$1 000,
- $K = 2$ and 20 customers of average value \$1 000;
- $K = 4$ and 40 customers of average value \$500; and
- $K = 6$ and 60 customers of average value \$333



→ The symmetric equilibrium offer distribution is uniformly random
→ It is independent on the number of competing marketing campaigns.

"Advertising Competitions in Social Networks," Masucci and S., Proc. of the American Control Conference (ACC), Seattle, WA, USA, May 24-26, 2017.

Conclusions

- We studied the advertising competition of two and also several marketing campaigns
  - We determined the optimal resource allocation to the potential costumers for the case of two marketing campaigns
- We provided:
  - The network value of a customer
  - The symmetric equilibrium offer strategy from which no campaign has interest to deviate
  - Some example scenarios to which our results can be applied

Extensions

- Asymmetric players and heterogeneous battlefields (work with Quan Dong Vu and Patrick Loiseau)

Thank you for your attention !

Thank you for your attention !
Questions?