A Framework for Optimal Investment Strategies for Competing Camps in a Social Network

PGMO Days 2017

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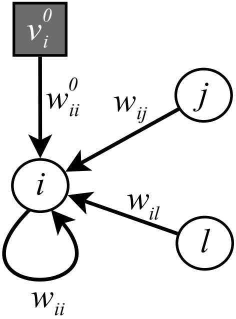
Introduction

Paper on arXiv - Good versus Evil: A Framework for Optimal Investment Strategies for Competing Camps in a Social Network

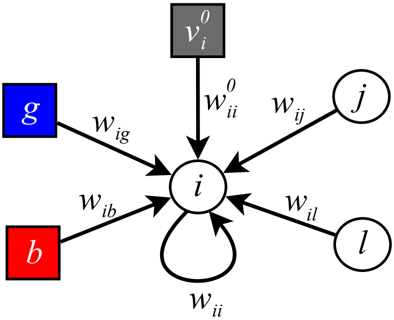
Friedkin Johnsen Model of Opinion Dynamics



$$v_i \longleftarrow \underbrace{w_{ii}^0 v_i^0}_{\text{bias}} + \underbrace{w_{ii} v_i}_{\text{self}} + \underbrace{\sum_{j \neq i} w_{ij} v_j}_{\text{network}}$$

Some Examples

1. Unbounded opinion values ($v_i \in \mathbb{R}$)
   - Fund collection
   - Sensors
2. Bounded opinion values ($v_i \in [-1, +1]$)
   - Elections
   - Product adoption



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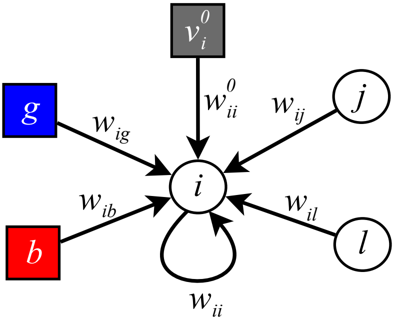
$x_i =$ investment by good camp on i

$y_i =$ investment by bad camp on i



$$\max_{\mathbf{x}} \min_{\mathbf{y}} \sum_i v_i$$

$$\text{s.t.} \quad \sum_i x_i \le k_g, \sum_i y_i \le k_b$$

$$x_i, y_i \le 1 \text{ (if bounded)}$$

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Convergence of Opinion Values and Computing $\sum_i v_i$

$$v_i^{\langle \tau \rangle} = w_{ii}^0 v_i^0 + w_{ii} v_i^{\langle \tau-1 \rangle} + \sum_{j \neq i} w_{ij} v_j^{\langle \tau-1 \rangle} + w_{ig} x_i - w_{ib} y_i$$

$$\mathbf{v}^{\langle \tau \rangle} = \mathbf{w}\mathbf{v}^{\langle \tau-1 \rangle} + \mathbf{w^0} \circ \mathbf{v^0} + \mathbf{w_g} \circ \mathbf{x} - \mathbf{w_b} \circ \mathbf{y}$$

$$\mathbf{v}^{\langle \tau \rangle} = \lim_{\eta \to \infty} \mathbf{w}^{\eta} \mathbf{v}^{\langle \tau-\eta \rangle} + \Big(\sum_{\eta=0}^{\infty} \mathbf{w}^{\eta} \Big) \big(\mathbf{w^0} \circ \mathbf{v} + \mathbf{w_g} \circ \mathbf{x} - \mathbf{w_b} \circ \mathbf{y} \big)$$

$$\mathbf{v} = (\mathbf{I} - \mathbf{w})^{-1} (\mathbf{w^0} \circ \mathbf{v^0} + \mathbf{w_g} \circ \mathbf{x} - \mathbf{w_b} \circ \mathbf{y})$$

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$$\mathbf{1}^T\mathbf{v} = \mathbf{1}^T(\mathbf{I} - \mathbf{w})^{-1}(\mathbf{w^0} \circ \mathbf{v^0} + \mathbf{w_g} \circ \mathbf{x} - \mathbf{w_b} \circ \mathbf{y})$$

$$\mathbf{r} = \left(\mathbf{I} - \mathbf{w}^T\right)^{-1}\mathbf{1}$$

Convergence of Opinion Values and Computing $\sum_i v_i$

$$v_i^{\langle\tau\rangle} = w_{ii}^0 v_i^0 + w_{ii} v_i^{\langle\tau-1\rangle} + \sum_{j\neq i} w_{ij} v_j^{\langle\tau-1\rangle} + w_{ig} x_i - w_{ib} y_i$$

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$$\mathbf{r} = \left(\mathbf{I} - \mathbf{w}^T\right)^{-1}\mathbf{1}$$

$$\sum_i v_i = \sum_i r_i w_{ii}^0 v_i^0 + \sum_i r_i w_{ig} x_i - \sum_i r_i w_{ib} y_i$$

Katz centrality of node i is the i^{th} component of $\left(\mathbf{I} - \alpha\mathcal{A}^T\right)^{-1}\mathbf{1}$

Result for Weighted Cascade-like Models

Proposition

Let $N_i = \{j : w_{ij} \neq 0\}$, $d_i = |N_i|$, and $j \in N_i \iff i \in N_j$. If $\forall i, w_{ig} = w_{ib} = w_{ii}^0 = \frac{1}{\alpha+d_i} = w_{ij}, \forall j \in N_i$, where $\alpha > 0$, then $r_i w_{ig} = r_i w_{ib} = \frac{1}{\alpha}, \forall i$.

$$\left(\mathbf{I} - \mathbf{w}^T\right)\mathbf{r} = \mathbf{1}$$

$$\iff \mathbf{r} = \mathbf{1} + \mathbf{w}^T\mathbf{r}$$

$$\iff r_i = 1 + \sum_{j \in N_i} w_{ji} r_j = 1 + \sum_{j \in N_i} \left(\frac{1}{\alpha + d_j}\right) r_j$$

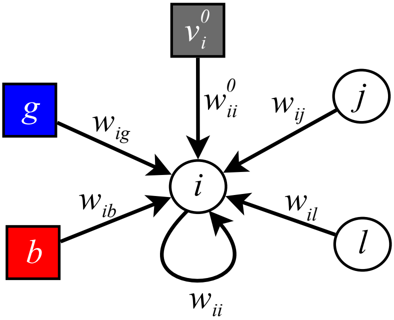
Let us assume that $r_i = \gamma(\alpha + d_i)$, where γ is some constant.

$$\gamma(\alpha + d_i) = 1 + \sum_{j \in N_i} \gamma = 1 + \gamma d_i$$

$$\iff \gamma = \frac{1}{\alpha}$$

So we have

Model of Opinion Dynamics (Concave Influence Function)



$$\max_{\mathbf{x}} \min_{\mathbf{y}} \sum_i v_i$$

$$\text{s.t.} \quad \sum_i x_i \leq k_g, \sum_i y_i \leq k_b$$

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x_i = investment by good camp on i

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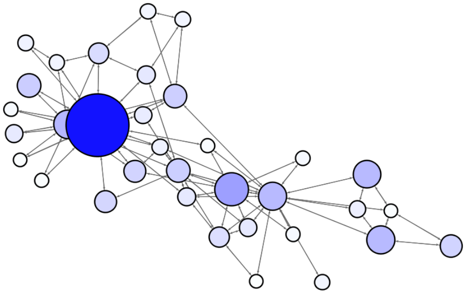
Investment Strategies (Concave Influence Function)

$p \downarrow$

$x_i \propto (r_i w_{ig})^{\frac{p}{p-1}}$



$p \uparrow$

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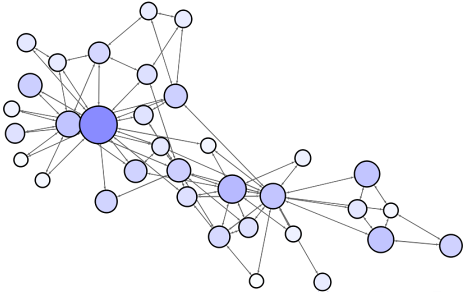

Bounded Investment Per Node (Concave) ($\forall i : x_i, y_i \leq 1$)

Proposition

Let $\hat{\gamma} > 0$ be the solution of

$$\sum_{i: r_i w_{ig} \in (0, t\gamma]} \left(\frac{r_i w_{ig}}{t\gamma} \right)^{\frac{t}{t-1}} + \sum_{i: r_i w_{ig} > t\gamma} 1 = k_g$$

$$x_i^* = 0, \text{ if } r_i w_{ig} \leq 0$$
$$x_i^* = 1, \text{ if } r_i w_{ig} > t\hat{\gamma}$$
$$x_i^* = \left(k_g - \sum_{i: r_i w_{ig} > t\hat{\gamma}} 1 \right) \left(\frac{(r_i w_{ig})^{\frac{t}{t-1}}}{\sum_{i: r_i w_{ig} \in (0, t\hat{\gamma}]} (r_i w_{ig})^{\frac{t}{t-1}}} \right), \quad \text{if } r_i w_{ig} \in (0, t\hat{\gamma}]$$

If there does not exist a $\hat{\gamma} > 0$, invest 1 on all nodes with $r_i w_{ig} > 0$ and 0 on all other nodes. The optimal strategy of the bad camp is analogous.

Bounded Investment Per Node (Concave) ($\forall i : x_i, y_i \leq 1$)

Trial-and-error Iterative Process

Until $x_i^* \leq 1, \forall i$

1. Use the optimal strategy for the unbounded case
2. If for any i, we get $x_i^* > 1$, assign $x_i^* = 1$ to node i with the highest value of $r_i w_{ig}$
3. Exclude node i and decrement the available budget by 1

Decision Under Uncertainty

The good camp plays first with uncertain information regarding w_{ig}, w_{ib}, w_{ii}^0, while the bad camp plays second

$$\max_{\substack{\sum x_i \leq k_g \\ x_i \geq 0}} \min_{E\mathbf{u} \leq f} \min_{\substack{\sum y_i \leq k_b \\ y_i \geq 0}} \sum_i r_i w_{ig} x_i + \sum_i r_i w_{ii}^0 v_i^0 - \sum_i r_i w_{ib} y_i$$

Common Coupled Constraints ($\forall i : x_i + y_i \leq 1$)

$$\max_{\hat{j}} \max_{\mathbf{x}} \sum_i x_i(r_i w_{ig} + \max\{r_i w_{ib} - r_{\hat{j}} w_{\hat{j}b}, 0\}) - \sum_i \max\{r_i w_{ib} - r_{\hat{j}} w_{\hat{j}b}, 0\} - r_{\hat{j}} w_{\hat{j}b} k_b$$

- Good camp chooses nodes with not only good values of $r_i w_{ig}$, but also good values of $r_i w_{ib}$
- Node $\hat{j}$ can be viewed as the node beyond which the bad camp does not invest on, as per its preference ordering
- If the good camp does not invest on node i (that is preferred by bad camp over $\hat{j}$), the bad camp would benefit $r_i w_{ib} - r_{\hat{j}} w_{\hat{j}b}$

Maxmin versus Minmax ($\forall i : x_i + y_i \leq 1$)

$$\max_{0 \leq x \leq 1} \min_{0 \leq y \leq 1-x} \sum_i v_i \geq \max_{0 \leq x \leq 1} \min_{0 \leq y \leq 1} \sum_i v_i$$

$$\max_{0 \leq x \leq 1} \min_{0 \leq y \leq 1} \sum_i v_i = \min_{0 \leq y \leq 1} \max_{0 \leq x \leq 1} \sum_i v_i$$

$$\min_{0 \leq y \leq 1} \max_{0 \leq x \leq 1} \sum_i v_i \geq \min_{0 \leq y \leq 1} \max_{0 \leq x \leq 1-y} \sum_i v_i$$

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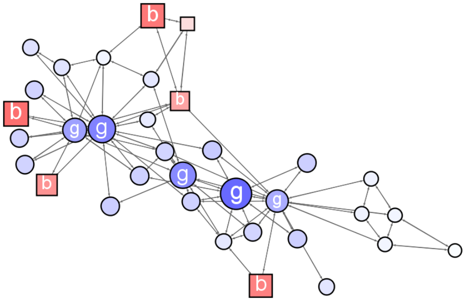
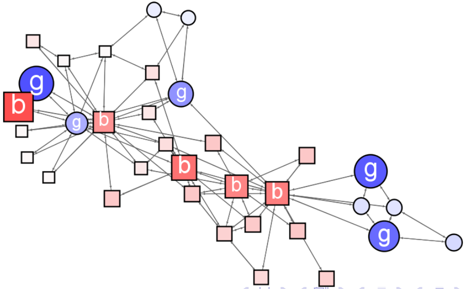
Common Coupled Constraints ($\forall i : x_i + y_i \leq 1$)

$\max_{\mathbf{x}} \min_{\mathbf{y}} \sum_i v_i$



$\min_{\mathbf{y}} \max_{\mathbf{x}} \sum_i v_i$



Two Phase Opinion Dynamics

$$\max_{x^{(1)}} \min_{y^{(1)}} \max_{x^{(2)}} \min_{y^{(2)}} \sum_i v_i^{(2)}$$

$$\text{s.t.} \quad \sum_i \left(x_i^{(1)} + x_i^{(2)} \right) \leq k_g \ , \ \sum_i \left(y_i^{(1)} + y_i^{(2)} \right) \leq k_b$$

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Let $\Delta = (\mathbf{I} - \mathbf{w})^{-1}$, then $r_i = \sum_j \Delta_{ji}$ and $s_i = \sum_j r_j w_{jj}^0 \Delta_{ji}$

$$\sum_i v_i^{(2)} = \sum_i s_i w_{ii}^0 v_i^0 + \sum_i \left(s_i w_{ig} x_i^{(1)} + r_i w_{ig} x_i^{(2)} \right) - \sum_i \left(s_i w_{ib} y_i^{(1)} + r_i w_{ib} y_i^{(2)} \right)$$

Two Phase Opinion Dynamics

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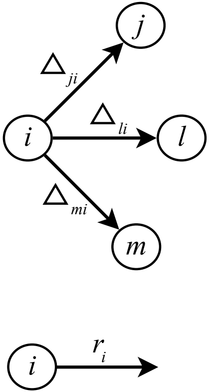
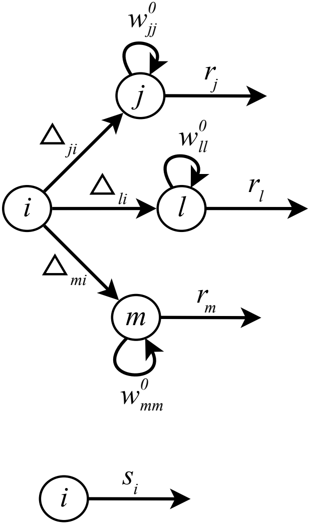
Loss to the bad camp if it acts myopically

$$k_b \left(\max_i \left(\max \{ s_i w_{ib}, r_i w_{ib}, 0 \} \right) - \max \left\{ s_{\hat{i}} w_{\hat{i}b}, 0 \right\} \right)$$

$$\text{where } \hat{i} = \arg\max_i r_i w_{ib}$$

Two Phase Version of Katz Centrality

$r_i = \sum_j \Delta_{ji}$	$s_i = \sum_j r_j w_{jj}^0 \Delta_{ji}$
	

Dependency between Parameters (One Camp, Unbounded)

Let $w_{ig} + w_{ib}$ be a constant θ_i

$$w_{ig} = \theta_i \left(\frac{1 + w_{ii}^0 v_i^0}{2} \right)$$
$$w_{ib} = \theta_i \left(\frac{1 - w_{ii}^0 v_i^0}{2} \right)$$

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Claim

- It is optimal to either exhaust the entire budget ($k_g^{(1)} + k_g^{(2)} = k_g$) or not invest at all ($k_g^{(1)} = k_g^{(2)} = 0$)
- It is an optimal strategy to invest on at most one node in a given phase

Opinion Dynamics in Two Phases

Let $b_{ji} = r_j w_{jj}^0 \Delta_{ji}$ and $c_i = w_{ii}^0 v_i^0$

For each candidate optimal pair (i^*, j^*) including $(0, 0)$

- If $-\theta_{i^*}\theta_{j^*} b_{j^* i^*}(c_{i^*} + 1) < 0$,
$$k_g^{(1)} = \min\left\{\max\left\{\frac{k_g}{2} + \frac{s_i^*}{\theta_{j^*} b_{j^* i^*}} - \frac{\sum_i b_{j^* i} c_i + r_{j^*}}{\theta_{i^*} b_{j^* i^*}(c_{i^*} + 1)}, 0\right\}, k_g\right\}$$
- If $-\theta_{i^*}\theta_{j^*} b_{j^* i^*}(c_{i^*} + 1) > 0$, then $k_g^{(1)} = 0$ or k_g

Opinion Dynamics in Two Phases

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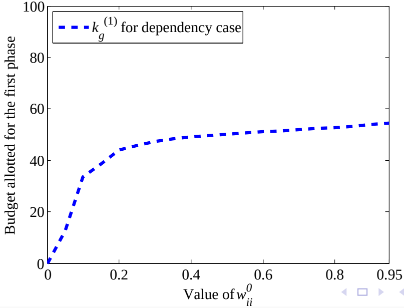

The Case of Competing Camps

Under practically reasonable assumptions

$w_{ij} \geq 0, \forall (i,j)$

$w_{ii}^0 \geq 0, \theta_i \geq 0, v_i^0 \in [-1,1], \forall i$

- transform the problem into a two-player zero-sum game with each player having (n^2+1) pure strategies
- show how the players' utilities can be computed for each strategy profile
- show existence of Nash equilibria and that they can be found efficiently using linear programming

Thank you!

Paper on arXiv - Good versus Evil: A Framework for Optimal Investment Strategies for Competing Camps in a Social Network