Aggregation in Mean Field Games

Daniela Tonon
Joint work with Marco Cirant

CEREMADE, Université Paris-Dauphine, France

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Mean field games were introduced by Lasry and Lions and by Huang, Caines and Malhamé in 2006 to describe Nash equilibria in differential games with infinitely many players

MFG theory analyzes :

- optimal control problems
- with (infinitely) many identical controllers/players

Typical features of the model :

- players act according to the same principles (they are indistinguishable and have the same optimization criteria)
- players have individually a minor (infinitesimal) influence, but their strategy takes into account the mass of co-players

Motivations :

- Problems arising in economy :
  - financial markets (Price formation and dynamic equilibria, Formation of volatility) (Lasry, Lions, 2006)
  - general economic equilibrium with rational expectations (Guéant, Lasry, and Lions, 2011)
- Dynamics of population models :
  - crowd motion : mexican wave "la ola", ... (Guéant, Lasry, Lions - Lachapelle, ...)
  - researchers' academic behavior (Besancenot, Courtault, El Dika...)
- Engineering literature :
  - large population stochastic wireless power control problem (Huang, Caines, Malhamé, 2003, Mériaux, Lasaulce...)
  - oil production (Guéant, Lasry, Lions, 2010)

The following is a simple form of MFG system with unknown (u, m)

$$\begin{cases} -u_t - \Delta u + H(x, \nabla u) = f(x, m) & \text{in } \mathbb{T}^N \times [0, T] \\ m_t - \Delta m - \operatorname{div}(m \nabla_p H(x, \nabla u)) = 0 & \text{in } \mathbb{T}^N \times [0, T] \\ m(x, 0) = m_0(x),\ u(x, T) = u_T(x) & \text{on } \mathbb{T}^N \end{cases}$$

where

- $H : \mathbb{T}^N \times \mathbb{R}^N \to \mathbb{R}$ is called the Hamiltonian, it is usually taken convex in the second variable
- $f : \mathbb{T}^N \times \mathbb{P}(\mathbb{T}^N \times [0, +\infty)) \to [0, +\infty)$ is the coupling
- $m_0 : \mathbb{T}^N \to \mathbb{R}^+$ is a probability density on $\mathbb{T}^N$
- $u_T : \mathbb{T}^N \to \mathbb{R}$ is a given function

To avoid boundary issues and exploit the compactness of the state space we set our problem on $\mathbb{T}^N$

Optimal control problem interpretation

$u(x,t)$ is the value function

Stochastic dynamics : $dX_s = v_s ds + \sqrt{2} dB_s$,
where (v_s) is the control, (B_s) is a Brownian motion

Cost : Let L be the Legendre transform of H

$$\mathbb{E}\left[\int_0^T L(X_s, -v_s) + f(X_s, m(X_s, s))\, ds + u_T(X_T)\right]$$

$\forall t \in [0, T]$ $m(x,t)$ denotes the density of population of small players at position x

the optimal control is formally given by the feedback $(t,x) \to -\nabla_p H(x, \nabla u(x,t))$

the second equation in (MFG) is the Kolmogorov equation of the process (X_s) when the small player plays in an optimal way

Monotone increasing coupling case

Motivation : model agents that prefer sparsely populated areas

In the MFG systems with uniformly parabolic diffusions we have $\exists!$ of smooth solutions, at least if the coupling is nonlocal and regularizing or if it has a “small growth" (Fixed point Theorem)

Analysis by PDE methods : see Cardaliaguet, Lasry, Lions and Porretta (2012), Lasry and Lions (2006, 2007), Gomes, Pimentel and Sánchez-Morgado (2013, 2014), Lions (2011), Porretta (2013)

Analysis by stochastic techniques : see Carmona and Delarue (2013), Huang, Malhamé and Caines (2006)

The case of couplings with an arbitrary growth has been discussed in Cardaliaguet, Lasry, Lions and Porretta (2012) (for quadratic hamiltonians, bounded by below couplings and smooth solutions) and in Porretta (2013) (for more general hamiltonians, bounded by below couplings but weak solutions)

Monotone decreasing coupling case

Motivation : study aggregation phenomena, i.e. when agents aim at converging to a common state

In 2009 Guéant considered simple population models where individuals have preferences about resembling to each other

Very few results exist in this direction and they only deal with very particular case. See

- Gomes, Nurbekyan, Prazeres 2016 and Guéant 2012 for the quadratic case
- Bardi, Priuli 2014 for the linear-quadratic case

Our setting

Let $Q = \mathbb{T}^N \times (0, T)$, we consider MFG systems of the form

$$\begin{cases} -u_t - \Delta u + H(\nabla u) = -f(x, m(x,t)), & \text{in } Q, \\ m_t - \Delta m - \operatorname{div}(\nabla H(\nabla u)\, m) = 0 & \text{in } Q, \\ m(x,0) = m_0(x), \quad u(x,T) = u_T(x) & \text{on } \mathbb{T}^N \end{cases}$$

where $\int_{\mathbb{T}^N} m_0\, dx = 1,\ m_0 > 0,\ m_0, u_T \in C^2(\mathbb{T}^N)$

Hypotheses

The Hamiltonian $H : \mathbb{R}^N \to \mathbb{R}$ is convex, $C^3(\mathbb{R}^N)$, and has a superlinear growth: $\exists \gamma > 1,\ C_H > 0$ s.t.

$$C_H^{-1}|p|^\gamma \leq H(p) \leq C_H(|p|^\gamma + 1),$$

for all $p \in \mathbb{R}^N$

Its Legendre transform, $L(q) := \sup_{p \in \mathbb{R}^N}\{p \cdot q - H(p)\}$ satisfies for some $C_L > 0$,

$$C_L^{-1}|q|^{\gamma'} - C_L \leq L(q) \leq C_L(|q|^{\gamma'} + 1),$$

for all $q \in \mathbb{R}^N$, where γ' is the conjugate exponent of γ, i.e. $\frac{1}{\gamma} + \frac{1}{\gamma'} = 1$

Hypotheses

The local coupling $f : \mathbb{T}^N \times [0, +\infty) \to \mathbb{R}$, $f \geq 0$ is continuous in both variables, differentiable w.r.t. the second variable and satisfies, for $\alpha > 0$,

$$|\partial_m f(x,m)| \leq c_f (m+1)^{\alpha-1} \quad \forall (x,m) \in \mathbb{T}^N \times [0,+\infty)$$

Hence, for all $(x, m) \in \mathbb{T}^N \times [0, +\infty)$

$$0 \leq f(x,m) \leq \frac{c_f}{\alpha}(m+1)^\alpha - \frac{c_f}{\alpha} + f(0) \leq C_f(m^\alpha + 1)$$

REM f is not necessarily increasing nor bounded from above
Let

$$F(x,m) = \int_0^m f(x,\sigma)d\sigma \quad \text{for } (x,m) \in \mathbb{T}^N \times [0,+\infty)$$

and $F(x, m) = +\infty$ otherwise
Then, for all $m \geq 0$ and a $C_F > 0$,

$$0 \leq F(x,m) \leq C_F(m^{\alpha+1} + 1)$$

Stationary case: Cirant 2016

Let $\Omega \subset \mathbb{R}^N$, we consider the following stationary MFG system

$$\begin{cases} -\Delta u + H(\nabla u) + \lambda = -f(m(x)), & \text{in } \Omega, \\ -\Delta m - \operatorname{div}(\nabla H(\nabla u)\, m) = 0 & \text{in } \Omega, \\ \int_\Omega m(x)dx = 1, m > 0 & \text{on } \Omega \end{cases}$$

Energy functional : for (m, w) s.t. $-\Delta m - \operatorname{div}(w) = 0$, $\int_\Omega m(x)dx = 1, m > 0$

$$\mathcal{E}_S(m, w) := \int_\Omega mL\left(-\frac{w}{m}\right) - F(m)\, dx$$

$$\mathcal{E}_S(m, w) \geq C_L \int_\Omega \left|\frac{w}{m}\right|^{\gamma'} m\, dx - C_F \int_\Omega m^{\alpha+1}\, dx$$

Stationary case: Cirant 2016

- If $\alpha < \gamma'/N$ $\exists$ a classical solution
  by means of a control of the "energy" associated to the system In other words, the decaying of the coupling is well compensated by the regularising properties of the diffusion

- If $\gamma'/N \leq \alpha < \gamma'/(N-\gamma')$ $\exists$ a classical solution only under additional assumptions on the coupling
  The control turns out to be more delicate, as aggregation may become the leading effect

- If $\alpha > \gamma'/(N-\gamma')$ classical solutions may not exist
  i.e. concentration due to the fast decay of the coupling cannot be compensated by the diffusion

Variational formulation for the evolutionary case

Let $\mathcal{K} \subset L^{\alpha+1}(Q) \times L^1(Q)$ be the set of pairs (m, w) satisfying

$$|w|^{\gamma'} m^{1-\gamma'} \in L^1(Q), m \geq 0$$

and $\forall \varphi \in C_0^\infty(\mathbb{T}^N \times [0, T))$

$$\int_Q (m\varphi_t + w \cdot \nabla\varphi - \nabla m \cdot \nabla\varphi) dxdt + \int_{\mathbb{T}^N} m_0(x)\varphi(x, 0)dx = 0$$

Energy functional $\mathcal{E}$ defined on $\mathcal{K}$

$$\mathcal{E}(m, w) := \int_Q mL\left(-\frac{w}{m}\right) - F(x, m)\, dxdt + \int_{\mathbb{T}^N} u_T(x)m(x, T)\, dx$$

REM : $\mathcal{E}(m, w)$ is not convex

Analysis of the Energy functional

If $\alpha < \gamma'/N$, the energy is bounded by below $\implies \exists c \in \mathbb{R}$ s.t.

$$c = \inf_{(m,w)\in\mathcal{K}} \mathcal{E}(m,w)$$

Let $(m^n, w^n) \in \mathcal{K}$ be s.t.

$$\mathcal{E}(m^n, w^n) \xrightarrow{n\to\infty} c$$

Then, up to subsequences, for $p = \alpha + 1$,

- $m^n \to \bar{m}$ strongly in $L^p(Q)$
- $w^n \rightharpoonup \bar{w}$ weakly in $L^{\frac{\gamma' p}{\gamma'+p-1}}(Q)$,

hence up to subsequences, $m^n \to \bar{m}$ a.e. in Q

Moreover, if $\gamma' > N+2$, then $m^n \to \bar{m}$ in $C^{0,\theta}(Q)$ for some $\theta > 0$

Then, the couple $(\bar{m}, \bar{w})$ is a minimiser of $\mathcal{E}$ in $\mathcal{K}$

How to link the minimiser $(\bar{m}, \bar{w})$ to a solution to the MFG system?

Idea : convexify the energy

$$\overline{\mathcal{E}}(m, w) = \mathcal{E}(m, w) + \int_Q G(x, t, m) dxdt,$$

where

$$G(x, t, m) := \frac{c_f + 1}{\alpha(\alpha + 1)} \left[(m + 1)^{\alpha+1} - (\bar{m}(x, t) + 1)^{\alpha+1} \right] - \frac{c_f + 1}{\alpha} (\bar{m}(x, t) + 1)^{\alpha} (m - \bar{m}(x, t))$$

NOTE: $G(x, t, m) \geq 0$ and $G(x, t, \bar{m}) = 0$

We consider now the MFG system associated to the convexified energy $\overline{\mathcal{E}}$:

$$\begin{cases} -u_t - \Delta u + H(\nabla u) = -f(x,m) + g(x,t,m) =: \tilde{f}(x,t,m), & \text{in } Q, \\ m_t - \Delta m - \operatorname{div}(\nabla H(\nabla u)\, m) = 0 & \text{in } Q, \\ m(x,0) = m_0(x), \quad u(x,T) = u_T(x) & \text{on } \mathbb{T}^N \end{cases}$$

IDEA : This is a particular case of Cardaliaguet, Graber, Porretta and T. (2015), which uses convex optimization methods from optimal transport problems (see Benamou and Brenier (2000), Cardaliaguet, Carlier and Nazaret (2012))

Proposition

If $\gamma' > 1$*, there exists a weak solution* $(\tilde{u}, \tilde{m}) \in L^q(Q) \times L^{\alpha+1}(Q)$ *to the monotone increasing MFG system,*
s.t. $(\tilde{m}, \tilde{m}\nabla H(\nabla \tilde{u}))$ *is a minimiser of the (convex) energy functional* $\overline{\mathcal{E}}$ *in* $\mathcal{K}$*.*

If $\gamma' > N + 2$*,* $(\tilde{u}, \tilde{m})$ *can be shown to be a classical solution. (Recall that in this case* $\bar{m} \in C^{0,\theta}(Q)$*)*

Let $q := \frac{\gamma(\alpha+1)(1+N)}{\alpha N-\gamma}$ if $\alpha > \frac{\gamma}{N}$, $q := +\infty$ if $\alpha < \frac{\gamma}{N}$

$(u, m) \in L^q(Q) \times L^{\alpha+1}(Q)$ is a WEAK SOLUTION to the MFG system if

(i) $\nabla u \in L^\gamma(Q)$, $mL(\nabla H(\nabla u)) \in L^1(Q)$ and $m\nabla H(\nabla u) \in L^1(Q)$

(ii) Hamilton-Jacobi inequality holds in the sense of distributions,

$$-u_t - \Delta u + H(\nabla u) \leq \tilde{f}(x, m) \quad \text{in} \quad Q,$$

with $u(\cdot, T) \leq u_T$,

(iii) The Fokker-Planck equation holds in the sense of distributions,

$$m_t - \Delta m - \text{div}(m\nabla H(\nabla u)) = 0 \;\; \text{in} \;\; Q, \quad m(0) = m_0$$

(iv) The following equality holds:

$$\int_Q m(x,t)(\tilde{f}(x, m(x,t)) + L(\nabla H(\nabla u)(x,t)))dxdt + \int_{\mathbb{T}^N} (m(x,T)u_T(x) - m_0(x)u(x,0))dx = 0$$

By strict convexity of $\overline{\mathcal{E}}$ we have $(\tilde{m}, \tilde{w}) = (\bar{m}, \bar{w})$ so that $(\bar{m}, \bar{m}\nabla H(\nabla \tilde{u}))$ is a minimiser of $\mathcal{E}$ in $\mathcal{K}$ and $(\tilde{u}, \bar{m})$ is a solution of the original MFG system

We have proved that when $\alpha < \frac{\gamma'}{N}$

Theorem

If $\gamma' > 1$, then, there exists a weak solution of the MFG system

$$(u, m) \in L^q(Q) \times L^{\alpha+1}(Q)$$

s.t. $(m, -m\nabla H(\nabla u))$ is a minimiser of $\mathcal{E}$.

If $\gamma' > N + 2$, then (u, m) is a classical solution

Can we say more on the regularity of solutions when $1 < \gamma' \leq N + 2$?

Idea : we can find smooth solutions through a penalisation argument under additional hypothesis on α

We consider the penalised Lagrangian

$$L_\eta(q) := L(q) + \frac{\eta}{N+3}|q|^{N+3}, \quad \forall q \in \mathbb{R}^N, \eta > 0,$$

and the associated functional defined on $\mathcal{K}$

$$\mathcal{E}_\eta(m, w) = \int_Q mL_\eta\left(-\frac{w}{m}\right) - F(x, m)\, dxdt + \int_{\mathbb{T}^N} u_T(x)m(x, T)\, dx$$

Then the system

$$\begin{cases} -u_t - \Delta u + H_\eta(\nabla u) = -f(x, m(x,t)), \\ m_t - \Delta m - \mathrm{div}(\nabla H_\eta(\nabla u)\, m) = 0 & \text{in } Q, \\ m(x,0) = m_0(x), \quad u(x,T) = u_T(x) & \text{on } \mathbb{T}^N \end{cases}$$

is s.t. $\gamma'_\eta = N + 3 > N + 2 \Longrightarrow \exists$ a classical solution (u_η, m_η) $\forall \eta > 0$

If

$$\alpha < \min\left\{ \frac{\gamma'}{N}, \frac{\gamma' - 2}{N + 2 - \gamma'} \right\}$$

using blow up techniques, there exists $C > 0$ s.t.

$$\|m_\eta\|_{L^\infty(Q)} \leq C \quad \forall \eta > 0$$

Then if $\gamma' > 2$ we improve the bounds on (u_η, m_η) in order to pass to classical limits

Theorem

Suppose that $2 < \gamma' \leq N + 2$ *and*

$$\alpha < \min\left\{\frac{\gamma'}{N}, \frac{\gamma' - 2}{N + 2 - \gamma'}\right\}.$$

Then, there exists a classical solution (u, m) *of the MFG system s.t.* $(m, -m\nabla H(\nabla u))$ *is a minimiser of* $\mathcal{E}$*.*

Conclusions and perspectives :

- Non uniqueness can be shown through examples: non minimizing solutions exist
- In the case $\gamma' = 2$ classical solutions could be found using the Hopf-Cole transformation
- What happens in the case $\alpha \geq \frac{\gamma'}{N}$? Mountain pass Theorem

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